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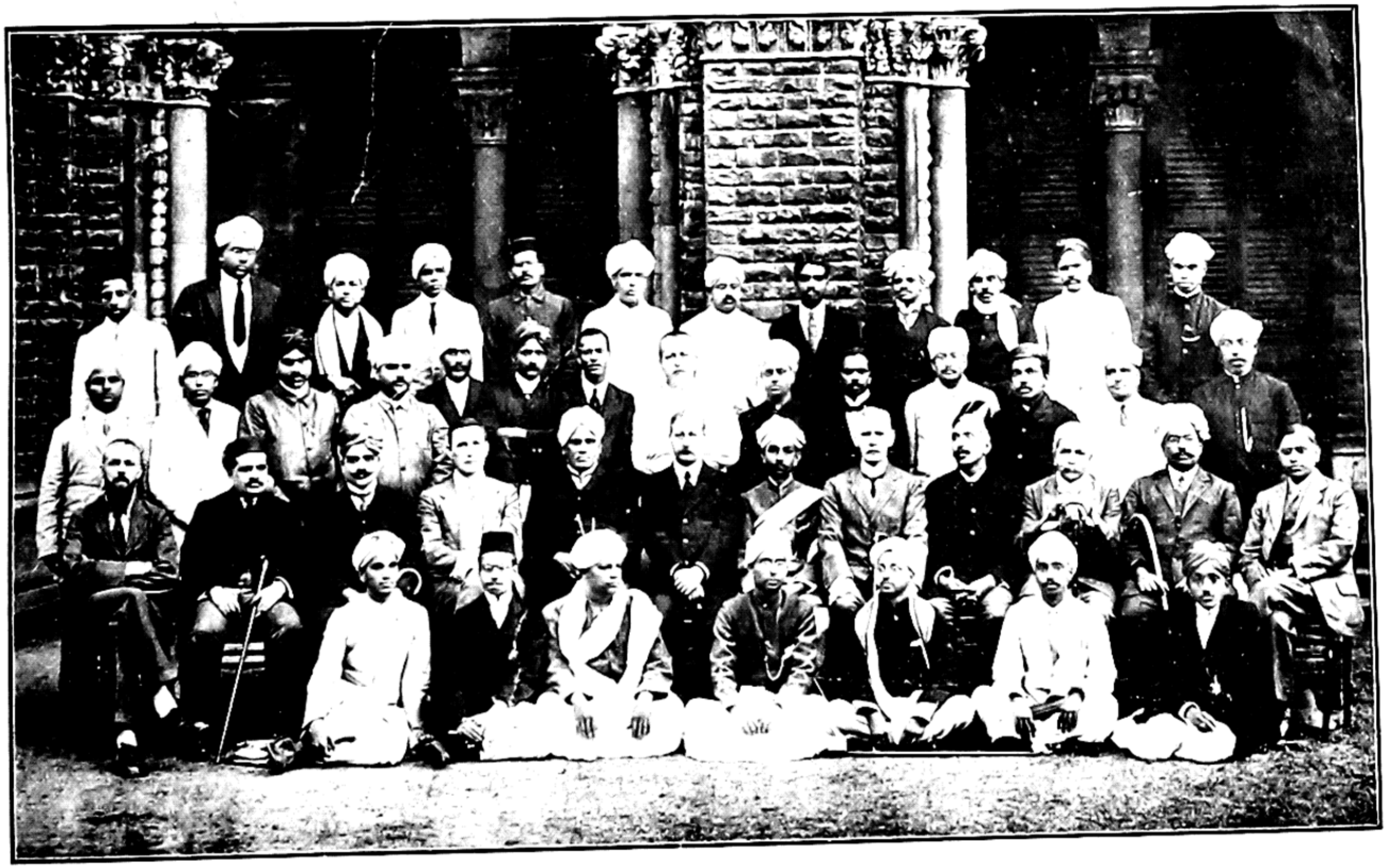
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INDIAN MATHEMATICAL SOCIETY

2ND CONFERENCE, JANUARY 1919, BOMBAY



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The Indian Mathematical Society.

SECOND CONFERENCE—11TH, 12TH & 13TH JANUARY 1919—BOMBAY

I. (Standing)—M. L. Chandratreya, C. N. Ganapati, R. Srinivasan, L. S. Vaidyanathan, K. N. Khandekar, S. R. Kashinatha Aiyar, G. G. Pendse, V. M. Gaitonde, K. B. Madhava, V. B. Naik, V. M. Datar, V. Gopalaswami Aiyangar

II. (Standing)—M. Vaidyanathan, S. R. Ranganathan, S. B. Belekar, K. D. Panday, Pandit Hemraj, V. D. Alurkar, C. P. Saldanha, Rev. Steichen, S. Narayanan, R. D. Karve, T. V. Mone, J. C. Swaminarayan, T. K. Venkataraman, M. Bhimasena Rao.

III. (Chair)—F. Hallberg, Justice Shah, R. P. Paranjpye, R. Littlehailes, V. Ramaswami Aiyar, A. C. L. Wilkinson, M. T. Naraniengar, Sir John Heaton, D. D. Kapadia, R. Ramachandra Rao, S. V. Ramamurthy, Balakram

IV. (Turf)—C. Krishnamachari, H. R. Kapadia, S. Narayana Aiyar, P. V. Seshu Aiyar, K. S. Patrachari, A. Narasingha Rao, T. M. Patil

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THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. XI.] FEBRUARY 1919. No. 1.

PROGRESS REPORT

The Second Conference of the Society was held in Bombay on the 11th, 12th and 13th of January 1919, when members from different parts of the country attended in fair numbers. It is for our members to judge how far this experiment of holding our Conference side by side with the Indian Science Congress was a success. A detailed account of the proceedings of the conference is given elsewhere in this journal.

2. The following gentlemen have been elected members of our Society:—

1. *Mr. N. B. Mitra, M.A.*—Professor of Mathematics, Ewing Christian College, Allahabad.
2. *Mr. S. A. Venkataraman, B.A.*—(Hon.) Student, Presidency College, Madras: (at concessional rate);
3. *Mr. Girdhari Lal Gupta, M.A.*—Professor of Mathematics, S. P. College, Srinagar (Kashmere);
4. *Mr. Jamshedji Edalji, B.A., B.Sc.*,--Professor of Mathematics; Gujarat College (Retired), 31/33 Karelwadi, Thakordwar, Bombay;
5. *Mr. Pestonji N. Wadia*—Secretary, B. J. Trust Funds Malabar Hill, Bombay;
6. *Mr. V. G. Dalvi, M.A. (Cantab), LL.B., Bar-at-Law*—73, Charni Road, Bombay;
7. *Mr. A. T. Banerji, M.A.*—Professor of Mathematics, Hindu College, Delhi;
8. *Mr. Ram Dass Nanda, M.A.*—Professor of Mathematics, Sanatan Dharma College, Lahore;

9. *Mr. F. V. Rushforth, B.A.* (Cantab),—Indian Finance Department, C/o Messrs. King & Co., Delhi.

10. *Mr. Royinton F. Fram Mirza* B. Eng. (Liverpool), A. M. I. C. E., Assoc. Royal T. C. M. Inst. M. E., Executive Engineer, Mysore Railway Staff, Bhatkal.

3. The General Body having voted upon, the following will be the re-constituted Committee for the years 1919 & 1920.

1. Professor A. C. L. Wilkinson, M.A.—*President.*
2. Mr. S. Narayana Aiyar, M. A. F. S. S.—*Treasurer.*
3. The Hon. Mr. R. P. Paranjpye, M. A., B. Sc.—*Librarian.*
4. Mr. Balakram, M.A., I. C. S.
5. Prof. Hemraj, M. A.
6. Prof. D. D. Kapadia, M. A., B. Sc.
7. Prof. V. B. Naik, M. A.
8. Prof. M. T. Naraniengar, M. A.
9. Mr. V. Ramesam, B. A., B. L.
10. Prof. E. B. Ross M. A.
11. Prof. P. V. Seshu Aiyar, B. A., L. T.
12. Prof. J. C. Swaminarayan, M. A.

4. The Committee take this opportunity to express the best thanks of the Society to Prof. R. N. Apte M.A., LL.B., F. R. A. S. who has now voluntarily retired from the Committee. Prof. Apte was one of the Foundation Members of the society and had been on the Committee from the very beginning and he was our President from 1912—1915. Our thanks are also due to Prof. Chowla, who retires from the Committee.

POONA,
15*th February*, 1919.

D. D. KAPADIA,
Hon. Joint Secretary.

Second Conference of the Indian Mathematical Society.

The second Conference of the Indian Mathematical Society was held in Bombay on the 11th, 12th and 13th January 1919—the opening session on the first day at the Sir Cawasji Jehangir University Convocation Hall, and the remaining sessions on the next two days at the Elphinstone College, Bombay.

All arrangements for a successful gathering had been completed by the Reception Committee under the able guidance of their Chairman, the Hon. Justice. Sir John J. Heaton, assisted by Prof. A. C. L. Wilkinson, the President of the Society, and the Hony. Jt. Secretary.

His Excellency Sir George Ambrose Lloyd, G.C.I.E., D.S.O., Governor of Bombay, had kindly consented to open the Conference; but owing to certain urgent and unforeseen circumstances His Excellency was prevented from doing so, and the Hon. Mr. C. Carmichael, C.S.I., I.C.S., Senior member of the Executive Council, Bombay, kindly opened the Conference in the name of His Excellency.

The Conference commenced its proceedings exactly at 11 a.m. on the first day, in the presence of a large gathering consisting of some 50 members of the Society who had come from various centres and a number of official and non-official gentlemen connected with the educational and other institutions of Bombay.

Sir John J. Heaton, Chairman of the Reception Committee, before delivering his Address of Welcome, explained briefly the unavoidable circumstances under which His Excellency was prevented from attending the conference, and expressed his great satisfaction that the Hon. Mr. Carmichael was going to open the Conference in His Excellency's absence. In his Address of Welcome* Sir John Heaton touched on a few important points relating to the constitution of the society.

The Hon. Mr. Carmichael, in opening the conference in the name of His Excellency, read out His Excellency's Message **which spoke in encouraging terms of the work done hitherto by the society.

The Hony. Joint Secretary (Prof. D. D. Kapadia) then read the Report† of the Society.

This was followed by the Presidential Address‡ by Prof. A. C. L. Wilkinson, which was listened to with great interest by the audience.

* Sir John J Heaton's Address of Welcome is given on p. 12 of this Journal.

** His Excellency's message is given on pp. 13–14.

† The Hon. Secretary's report is given on pp. 15–19.

‡ Prof. Wilkinson's Address is given on pp. 19–30.

On the motion of the Hon. Mr. R. P. Paranjpye, seconded by the Hon. Justice. L. A. Shah, a hearty vote of thanks was accorded to the Honorable Mr. Carmichael for presiding on the occasion and to the other visitors for their interest in the work of the Society. The formal proceedings of the conference terminated for the day at this stage, and an informal meeting of the members of the society was held thereafter under the Chairmanship of Prof. A. C. L. Wilkinson.

At this meeting Prof. Wilkinson drew the attention of the Society to the importance of doing some work on the lines of the reports submitted by the International Congress of Mathematicians held at Rome in 1912. While all the other countries were represented at that Congress, India was conspicuous by her absence. There was a great necessity of improving upon the kind of work in mathematics which we at present were doing in India. All useless work which principally involved nothing but the rule of thumb ought to be avoided. Elementary text-books ought to be written on the lines of the actual requirements of the class of students for whom they were meant, and it was high time that various Indian Provinces should come out of their old time-worn working grooves and shew their readiness to grasp new ideas and methods. The Indian Mathematical Society was the proper body, in his opinion, to initiate a movement of this kind.

This elicited a very interesting and prolonged discussion in which various members took part. The general trend of these discussions could be summarized as under—

That, after carefully going through the reports of the International Congress, points of inquiry should be first settled—such as (*a*) methods of examination, (*b*) methods of teaching, (*c*) text-books, etc.

That members in small committees in each Presidency should be made to undertake the inquiry and suggest the necessary improvements.

That though a complete systematic survey of the country might require vast expenditure and energy, it was an inquiry worth while going into.

That the work done by such small provincial committees should be collected by a General Committee and that the latter should submit their suggestions thereon.

Eventually, it was resolved to form two small committees as under to carry on inquiries and submit their suggestions—

Mr. M. T. Naraniengar,
Mr. R. Littlehailes,
Mr. P. V. Seshu Aiyar,
Mr. T. K. Venkataraman; } For Madras;

Mr. A. C. L. Wilkinson,
Mr. V. B. Naik,
Rev. A. Steichen. } For Bombay.

A similar committee for the Punjab is to be formed hereafter, after consulting the gentlemen concerned.

As Prof. Wilkinson was himself the originator of this discussion and was keenly interested in the investigation, it was deemed proper that he should be the Convener and direct the working of these committees.

After some discussion on some other minor items in connection with the society, the meeting dispersed at 2 p.m.

In the afternoon at 5 p.m., Sir John Heaton, Chairman of the Reception Committee, was "At Home" to all the members of the society, at his bungalow in Harkness Road, Malabar Hill. Some other distinguished visitors were also present and the guests after freely partaking of the bumper refreshments provided for the occasion and after spending about an hour and a half in pleasant chit-chat and mutual intercourse, returned to their homes. Sir John and Lady Heaton did their very best to make the gathering a success, and the guests thanked them most cordially for their hospitality.

On the second day, punctually at 8 a.m, commenced the more serious programme on the agenda of the Conference, under the Chairmanship of Prof. A. C. L. Wilkinson, in the Elphinstone College Buildings.

The First two papers mentioned below were read and discussed after which the Conference adjourned at 9 a.m. for a group-photo of the members of the society. This being over, the Conference resumed its work at 9-30 a.m., under the chairmanship of the Hon. Mr. R. P. Paranjpye, and the remaining papers fixed for the day were read and occasionally discussed. Then the Conference adjourned for the next day.

The papers presented on the day were as under:—

1. *On Cartesian Ovals*—by V. Ramaswami Aiyar, M.A.

This paper gives a more symmetrical treatment than usual of the Cartesian Oval, and proves several new metrical properties of the curve: this method of treatment leads to a remarkable expression for the radius of curvature at any point of the curve.

2. *On the Computation of the Ephemeris of a Planet and a Comet*—by R. N. Apte, M.A., LL.B., F.R.A.S.

This paper, after explaining the meanings of the ephemeris, the elements of parabolic and Elliptic orbits, positive and negative directions and time intervals, and the heliocentric equatorial co-ordinates, proceeds to the explanation and computation of—Gauss's constants, r and v in Elliptic and Parabolic orbits, heliocentric co-ordinates in terms of eccentric anamoly, etc., thus gradually leading to the different steps in the computation of an ephemeris with suitable illustrations.

3. *On a System of three directly similar triangles connected with a Triangle*—by S. Narayanan, B.A.

If A, B, C, D be any four given points in a plane, their pedal triangles, viz., of each point taken with respect to the triangle formed by the other three, are all similar, and have the same *double point* which is the centre of the rectangular hyperbola through A. B, C, D.

4. *Analytic Theory of Arithmetic Functions*—by F. Hallberg.

The paper separates the group of operations of ordinary algebra from the group of entities. For the latter is substituted a complex number system with an infinite number of units, when it is found that this, together with the group of operations, forms a logical system, if the process of multiplication is generalized in a way different from the ordinary one, not even the commutative law of multiplication being sacrificed.

5. *Multiplication of Infinite Integrals*—by K. B. Madhava, M.A.,

This paper investigates the conditions under which the product of two convergent infinite integrals

$$\int_0^\infty u\,(x)\,dx \text{ and } \int_0^\infty v\,(x)\,dx$$

can be expressed as a convergent integral in the form

$$\int_0^\infty \int_0^\infty u(y)\, v(x-y)\; dx\; dy,$$

the mode of forming the last integral being analogous to Cauchy's method of writing the typical term in the product of two infinite series. Incidentally a (small) o inequality in one of Hardy's papers is replaced by a (capital) O inequality.

6. *The Infinite Product* $\prod\left(1-\frac{x^{2n}}{r^{2n}}\right)$ —by S. R. Ranganathan, M.A.

The value of this product is deduced in a finite form, which holds good in the neighbourhood of $x=t$, but not at $x=t$, t being an integer. When $x=t$, the modified product $\prod_{r=1}^{\infty}{}'$ (the dash indicating the omission of the factor in which $r=t$) is evaluated. The logarithmic differentiation of the modified product leads to the values of $\sum \frac{}{r^{2n}-x^{2n}}$, which holds good for integral values of t. These and other allied formulae give neat results for certain values of x and n.

7. *An Extension of Feuerbach Theorem.*—by M. Bhimasena Rao.

The circle, passing through the points of contact of a conic inscribed in a triangle and having its centre at a given point P, is termed the *contact circle* of P with respect to the triangle A B C, and different cases of such contact circles are discussed. The paper concludes with the theorem that "the pedal circle (or contact circle) of any point with respect to a triangle cuts the nine-points circle at an angle equal to the sum of the angles at which it intersects the sides of the given triangle.

In the afternoon at 2 p. m., the members gathered together on the Apollo Bunder, where a fine big launch, so generously placed at the disposal of the society by Mr. G. W. Hatch, I.C.S., Chairman of the Bombay Port Trust, was in waiting to take them on an excursion in the Bombay Harbour. The visitors were then taken to the Elephanta Island, where they had an opportunity to survey the antiquarian relics carved on the scarps of mountains about the time of the revival of Brahmanism, after the decline of Buddhism in India. Refreshments were provided here for the visitors, and the members one and all enjoyed this excursion of about five hours.

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On the third and last day the adjourned sessions of the Conference commenced at 8 a. m., partly under the chairmanship of Professor A. C. L. Wilkinson and partly under the chairmanship of Mr. V. Ramaswami Aiyar, M.A. The following papers were successively read and discussed:—

8. *Fractional Differentiation*—by M. T. Naraniengar, M.A.

The notations $D^{\frac{1}{2}}$, $D^{\frac{1}{3}}$, ...$D^{\frac{1}{n}}$ are defined by the laws of combination subject to the index law and the following resnlts are obtained—

(i) $D^{\frac{1}{2}} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} x^{n-\frac{1}{2}}$;

(ii) $D^{\frac{1}{3}} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{3})} x^{n-\frac{1}{3}}$; and so on.

Inversely

(iii) $D^{-\frac{1}{2}} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} x^{n+\frac{1}{2}}$;

(iv) $D^{-\frac{1}{3}} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{4}{3})} x^{n+\frac{1}{3}}$, and so on.

These processes are applied to obtain some generalizations of well-known theorems—such as, Leibnitz's Theorem, Binomial Theorem, Exponential Theorem, Euler's Theorem on homogeneous functions and others.

9. *On a Fundamental property of the Elliptic Functions*—by A. C. L. Wilkinson M.A.

The paper states that an examination of the two periods of the Jacobi, Elliptic Functions, when k is complex, shows that an when defined in a certain way, the coefficient of i in their ratio is positive.

10. *On Cosmic Synthesis.*—by S. V. Ramamurthy I.C.S.

Starting from two hypotheses about the relation of time, space and matter that they are all one entity (mass) but are recognizable groups of it of different scales of size, various laws, now deduced experimentally, are deduced mathematically—*viz*, Law of Gravitation, Newton's Laws of Motion, Boyle's Law, Oersted's Law, Ohm's Law. Conceptions of polydimensional time and mass are introduced, and a solution of the problem of n bodies is given.

11. *Some Elementary Properties of a Tetrahedron.*—by Balak Ram, I.C.S.

In this note, the author obtains by Quaternions the conditions that two or more of the shortest distances between the opposite edges of a tetrahedron should intersect one another, and the conditions that any one of these shortest distances should intersect the line joining the middle points of any pair of opposite edges. Properties of the special tetrahedron possessing an orthocentre are also obtained.

12. *On certain Definite Integrals and Series connected with Bernoulli's Numbers*—by C. Krishnamachari, M.A.

Some definite integrals for B_n and E_n are obtained, and expressions for the trigonometrical functions are deduced as infinite integrals. Some applications of the theory of residues are illustrated and by examining the residues of functions with $\sin \pi z$, $\cos \pi z$, $\tan \pi z$ in the denominator, summation formulae are obtained.

13. *On Curves similar to their n^{th} Evolutes*—by A.Narasinga Rao, M.A.

The curve $s=f(\psi)$ is similar to its nth evolute, if $f^{n}(\psi)=\alpha f(\psi+\beta)$. The general solution of this fundamental equation gives $f(\psi)=\Sigma C e^{a\psi}\cos b\psi$, where $z=a+i\,b$, satisfies $z^n=\alpha e^{z\beta}$, which, besides including the spiral and cycloid family, leads to several new and interesting curves. The latter part of the paper deals with oblique evolutes and the determination of a curve which is its own nth evolute.

14. *Bernoulli's Polynomial and Fourier Series*—by S. R. Ranganathan, M.A.

In this note several formulæ are obtained from the principal Fourier series; also expressions are derived for certain series, such as

$$1-\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}-\frac{1}{8^{2n}}-\frac{1}{10^{2n}}+\ldots=\frac{\pi^{2n}(3^{2n}-1)(2^{2n-1}-1)}{3^{n}(2n)!}B_n.$$

15. *Conics with vanishing* Θ, Θ'—by R. Vythyanathaswamy.

Further applications of the author's "Cyclic centres' to properties of the polar line of a triangle with respect to a circumconic are given here.

16. *Legitimacy of the Ordinary Complex Number*—by Dr. G. A. Miller.

A group conception of the complex number in which, for multiplication, a complex number is represented by a linear group transformation $x'=ax-by$, $y'=bx+ay$.

At the conclusion of the reading of papers, the Hony. Jt. Secretary, explained briefly the circumstances which contributed to the success of this Second Conference of the Society. On behalf of the Reception Committee, he tendered a vote of thanks to all those who had taken the trouble to attend the Conference, even at certain inconvenience to themselves. He further added that on that very day, India had the peculiar good fortune to learn of a remarkable episode in the history of British Rule, *viz.*, the appointment of Sir S. P. Sinha as the Under-Secretary of State for India with a peerage. This he hoped would act as a powerful incentive to all loyal and patriotic sons of India, and would contribute to the material and intellectual advancement of India. The best thanks of our Society were due to the Vice-Chancellor and the Syndics of the University for the loan of their hall, to Principal Covernton of the Elphinstone College for the use of the College premises, to the Chairman of the Bombay Port Trust for the launch and the arrangements at the Docks, and to the Mint Master, Bombay, for their kind co-operation. He alone could personally appreciate what Sir John Heaton as the Chairman of the Reception Committee, and Prof. Wilkinson had done to make this conference a success. His best thanks were due to all of them—as well as to four members of our Reception Committee, Messrs. Datar, Chandratraya, H. R. Kapadia and Gaitonde-for all they had done in connection with the conference.

On the motion of the Hon. Mr. R. P. Paranjpye, seconded by Mr. V. Ramaswami Aiyar, it was resolved that

"This Conference of the Indian Mathematical Society takes this opportunity to put on record their deep sense of gratitude for the appointment of Sir S.P. Sinha, as the Under Secretary of State for India, with peerage, and that a copy of this resolution be transmitted to Sir S. P. Sinha".

The motion was carried with acclamation.

On the motion of the Hon. Mr. R. P. Paranjpye, seconded by the Hon. Mr. R. Littlehailes, it was resolved that

"A hearty vote of thanks of the society be conveyed to the Vice-Chancellor and the Syndics of the University, to the Principal, Elphinstone College, to the Chairman, Bombay Port Trust, and to others for their kind co-operation in the work of the Conference."

The motion was adopted unanimously.

While concluding the sessions of the Conference, the President, Prof. Wilkinson, offered a few suitable remarks, and expressed a wish that, circumstances permitting, they would try to meet for the third Conference at some central place which it would be too premature to notify at that stage.

With a vote of thanks to the Chairman, the Conference, concluded its sessions at 11-30 a.m.

In the afternoon at about 2 p.m., the members were taken for a visit to His Majesty's Mint. Here the visitors were shown the various departments of the Mint by the Mint authoritiss, and as soon as this was over, the whole party marched on to the Alexandra Docks. At the Docks, our members had the opportunity of witnessing the troops and Kut prisoners, who having disembarked but shortly before from a troop-ship from Basra were being entertained in the Dock-sheds. Some of our members were introduced to the General in charge of the arrangements and the scene there produced such a sympathetic impression on some of us, that a small sum of Rs. 25 was immediately subscribed by our members on the spot, and handed over to the Officer commanding as an humble contribution from the Indian Mathematical Society towards the entertainment fund.

After walking over the Docks, the whole party rested for a while at the "Mole" Railway Station, and there after some light refreshments our members assembled in a meeting under the chairmanship of Mr. V. Ramaswami Aiyar, and the following resolution, proposed by Mr. V. Ramaswami Aiyar and seconded by Mr. M. T. Naraniengar, and supported by Messrs. Hemraj, P. V, Seshu Aiyar and T. V. Mone, was unanimously adopted—

> "That the most cordial thanks of the Members of the Mathematical Conference belonging to outstations be conveyed to the Reception Committee, and to their Chairman, Justice Sir John Heaton for the kindness and attention extended to them during their stay in Bombay."

Mr. D. D. Kapadia, Hon. Jt. Secretary, thanked the members on behalf of the Reception Committee, for their kind appreciation of what little work they had done; and after some happy exchange of mutual greetings and farewell, the members dispersed at about 5 p.m. This concluded the Second Conference of the Indian Mathematical Society.

Address of Welcome by the Hon. Justice Sir John Heaton, Kt.

Your Excellency,

It is my privilege to offer you a welcome and also to offer you thanks on behalf of the Indian Mathematical Society for your kindness in consenting to open our Second Conference. We are neither an old, nor a powerful Society, but we represent a not unimportant effort after intellectual improvement and the spread of sound thinking in India. As such we venture to hope that our wishes will have some slight value and we do most genuinely wish for Your Excellency the greatest possible success in the outcome of your administration.

We value Your Excellency's presence amongst us not only on account of the high office you hold. There are other reasons also.

The work which Your Excellency had accomplished before taking up your present office demonstrates a capacity for accurately stating a problem and a realisation that the collection of correct data is an essential preliminary to the solution of a problem. In other words, Your Excellency has manifested the spirit of Science, and in particular of Mathematics—the earliest and perhaps the most important of Sciences. Moreover, Mathematics displays to a peculiar extent an intellectual method which is beginning to appeal in a marked degree to men of affairs. In Mathematics, a conclusion cannot claim, and will not be accorded, any importance until every process leading up to that conclusion is frankly displayed and is fully exposed to the most searching examination. That is the essence of the Mathematical method. It is easy to imagine the dismay with which old-fashioned politicians and diplomatists would regard such a method if applied to their branches of affairs. But it has now become apparent that a change is in progress in the minds of men of affairs. And in Your Excellency's interest in our Society we have an indication that such men no longer hold themselves aloof from the application to political and administrative problems of something of the spirit of the Science of Mathematics.

Our welcome and our thanks therefore are the expression of a feeling much deeper than that of mere formal compliment.

I now ask Your Excellency to be so good as to declare the Conference open.

Message of

His Excellency Sir George Lloyd D.S.O. K.C.I.E.,

Governor of Bombay.

Mr. Chairman and Gentlemen :—

First let me say how extremely sorry I am to have been under the unavoidable necessity of cancelling my engagement to open the Conference of the Indian Mathematical Society this morning. I have the greatest sympathy with the ideals of your Society and have been very interested to learn, from a perusal of the papers sent me, of the work it has done, of the Library and Journal it maintains and of the keenness which its members show in attending and writing papers for these conferences.

I fully recognise the difficulty which you experience in attracting members from all parts of such a vast country as this, but I am glad to see that Local Governments have contributed their quota of support in the shape of travelling allowances to such of their officers as attend the conference. I hope that the Society will not hesitate to appeal to my Government for any further assistance they may require and, while I cannot promise to grant its request, I can at least assure you that I will lend a sympathetic ear to any such suggestion.

I understand that the main aim of your Society is to encourage research in pure mathematics by post-graduates, and I know that most of you look upon this research as an end in itself; and upon the results of the research as so many valuable additions to the Science of pure mathematics. Now, while entirely sympathising with this point of view, I wish to point out to you that your research work has other and very valuable results.

Mathematics is an essential adjunct to almost every science known in the world; any work, therefore, which increases our knowledge of mathematics, tends to increase our knowledge, or our means of obtaining knowledge, in all these sciences. From this point of view it may be held to have a practical value for practical workers in the sister sciences. It might however seem that the elucidation of some of the abstruse and highly technical problems which I see down on your programme, is only very remotely connected with the consideration of practoal sciences, and from a narrow point of view, of course this is true; but I would like to call your attention to another side of the question. One of the dangers

of the existing Indian educational system is that it has a tendency to give undue importance to examinations rather than to original observation or independent thought. Research work is valuable not only in discovering new problems, but also in training workers to use the self-reliance and independence of thought they have gained to the problems of their every day life. Knowledge acquired, or studies pursued, for the sake of examinations alone are not of the highest value, but the kind of work which your Society fosters and encourages is, in my opinion, of the greatest utility not only by reason of the results obtained, or the encouragement to further studies it affords, but also because it strengthens the mind and character of those engaged in it, and it is mainly for that reason that I am so glad to see your Society prospering.

I see that you number among your members men of many professions—civilians, politicians and professors. Your first President, Mr. Hanumant Rao, was a Professor of the Engineering College, Madras, and your fourth, Dewan Bahadur Ramachandra Rao, was a Secretary to Government. Such men cannot but profit in their ordinary profession from the concentration and independence of thought they have had to employ in these mathematical research studies. As regards the quality of the mathematical work which Indians are now doing and the position of the scholars who are members of your Society, I need only mention one out of many names. Mr. Ramanujan, after a most brilliant career, starting from a very humble origin, has been elected a Fellow of the Royal Society, one of the highest honours a man can hold, and I wish him all success and honour in his future career.

Gentlemen, I must not detain you longer. Once again I must say how disappointed I am not to be present at your Conference. I wish it all success and I hope that you will enjoy and profit by the discussions I see on the programme and by the meeting and interchange of views between members from different parts of India which is one of the chief values of such conferences.

Report of the Hony. Jt. Secretary, Prof. D. D. Kapadia.

Your Excellency, and Gentlemen,

On behalf of the Committee of the Indian Mathematical Society, it is my very pleasant duty to extend to you all a very cordial and friendly welcome at this—the second Conference of the Indian Mathematical Society, and express our sense of appreciation for the trouble you have

taken to be present at the conference and make it a success. We consider ourselves extremely fortunate in having Your Excellency amongst us to open our Conference to-day; and the cheerfulness with which Your Excellency accepted our invitation is a clear mark of appreciation of the work done by our humble Society up to now in a quiet and unostentatious manner. It is useless for us to say that the Society feels highly flattered by your presence amongst us, and it shall be our constant endeavour to exert our utmost to prove ourselves worthy of this honour.

At our First Conference held at Madras in December 1916, I had the privilege of relating in detail the history of the foundation, the progress and the work done by our Society up to then. This has been printed in our Journal for February 1917. On this occasion, therefore, it is needless to repeat the whole thing over again.

Our Society was founded in April 1907 and, during these 12 years of our existence, we have been steadily progressing and growing in numbers. Our present strength is 197 members—viz: 1 patron, 1 Hon. member, 9 life members and 186 ordinary members. We have reason to believe that, as our Society becomes more widely known and our aims and objects more clearly understood, scholars will flock in greater numbers to our Society. But our first and foremost duty is to aim at greater efficiency; that means that we ought to be in a sounder financial position than what we enjoy at present and that we should have the most active co-operation of our members.

In connection with the first point we are sorry to say that most of the income earned by us as subscriptions is spent in the course of the ordinary routine work of the Society, and hardly any surplus is left for adopting means to increase our efficiency; and in connection with the second point, our work being a labour of love, it requires a certain amount of persuasion to make our members co-operate in the work of our Society.

Our Society is meant primarily for the promotion of mathematical study and research, and it therefore does not appeal very forcibly to all engaged in the teaching of elementary mathematics. We had once formed a plan of having a branch of our Society dealing with elementary mathematics, but the idea had to be kept in suspense for the simple reason that before the parent Society was established on a rigorous basis ensuring its permanance and stability, it was not advisable to fritter away our energy in different directions.

For achieving the high ideals of the promotion of mathematical study and research, the Society has organized- (i) a Journal for the mutual interchange of ideas and as a medium for inter-communication, (ii) a circulating Library having periodicals in sufficient numbers to facilitate quick circulation of them amongst our members, and (iii) a Library of reference books. We have at present a Circulating Library with some 35 Periodicals and a small Reference Library.

What we said in our last report in connection with our Journal and the Library, remains nearly true to-day. When we met last, we hoped to achieve some tangible results by canvassing financial assistance from all possible sources, such as the Home Government, the various Provincial Governments, and the Maharajas and Ruling Princes of India. But since then events took a dramatic turn, and the united and organized efforts of all lovers of Humanity and Freedom were required to be employed to save mankind from an enforced thraldom at the hands of a Nation, who up to then had achieved a name in the realm of Learning and Science, but who perversely thought that 'Might was Right,' and that 'the end justified the means.' During the course of the titanic struggle, which is now happily ended, there could not be but *one* and *only one* aim in view, namely: the vigorous prosecution of the righteous campaign without the least diversion of all possible available means and resources of the Empire. It is indeed a matter of pride for us to note that India had opportunities of playing her own part in the struggle and that she in general acquitted herself creditably. Our Committee therefore considered it advisable to postpone our efforts for the collection of funds until the events took a favourable turn.

Now, however, the circumstances have improved; the armistice has been concluded and very soon we shall be in a position to hear the joyful tidings of a Peace-ensuring, to a greater or less extent, the safety and liberties of mankind. People will have now leisure and energy to devote to their domestic problems, and our Committee hope to achieve some tangible results in future, by representing our real and legitimate wants to all those who are in a position to contribute their mite towards the promotion of the intellectual and material progress of India as a whole. We trust that our appeal will not be in vain and our Society will be able to do something soon for the advancement of the Science of Mathematics, the oldest of the Sciences, for which India had a peculiar fascination from the remote past.

Our Journal has generally been an attractive feature amongst our activities. There are several directions in which one would like to set improvements in the journal. We are however not far from correct, when we say that it has stimulated many young gentlemen of promise to contribute articles and notes on mathematical subjects, and problems. We need hardly confess that though we have not been as yet able to attain the standard of originality and excellence found in some European and American Journals, yet our Journal has sufficiently attracted notice from outside even at this early stage of its existence. Our energetic Editor Prof., M. T. Naraniengar has been chiefly responsible for its get-up and for its periodic appearance.

Our Circulating Library is a very useful appendage of our Society. We are getting some 35 European and American periodicals for circulation amongst our members. To make our Library serve its purpose more efficiently, we want several of these periodicals in duplicate and triplicate copies. Sometimes one hears of complaints from our members with limited means that the burden of return postage often precludes them from making a free use of the Library.

The stock of reference books and journals in our Library, suitable for advanced and research work, is very limited and scanty. Every year we have been trying our best to effect some improvement in this direction. But by the end of the year hardly any surplus remains to be utilized for buying good books. The late Sir Ratan Tata cheerfully gave us a donation of Rs. 500 in 1909. Beyond that we have not been fortunate to receive help from the public. The concentration of all books on Advanced Mathematics, and back numbers of important Mathematical Periodicals in a suitable Central Library is a great public desideratum, and the Library of our Society can aptly be made a *centre* for all kinds of Mathematical books both ancient and modern. It should be our constant aim to try to make our Library a source of inspiration to all workers in Mathematics.

Only but recently, H. E. Lord Willingdon's Government recognized the necessity of such a central Library for Oriental Literature by liberally contributing their mite towards the Bhandarkar Oriental Research Institute. Not only has the whole valuable collection of the rich Sanskrit Manuscripts of Deccan College, Poona, been handed over to the Governing Body of the Bhandarkar Institute, but also an annual recurring grant has been placed at the disposal of that Institute. Our wish therefore for centralization of our Library on similar lines, we presume, could not be considered as highly extravagant or coming

within the category of chimerical schemes impossible of execution. This question is not one of local importance alone. It affects various provinces equally, and hence all concerned in the intellectual welfare of India ought to participate in this matter.

Our Society is capable of immense potentialities. If we had sufficient funds at our disposal, we could organize schemes for offering prizes to Indian workers for some kind of original work in Mathematics, or even offer scholarships and fellowships to deserving students to enable them to concentrate their attention and dive deeper in their studies as Research Students. Our helplessness in this respect is almost criminal. When the famous mathematician, Mr. Ramanujan, F.R.S., sought some modest help from our Society to enable him to devote his attention to Mathematical studies and research, our bankruptcy was made manifest. We feel it imperative that our Society should spare no efforts to found a Research Fund for the promotion of Mathematical Research in India with the generous help of our Government, the Ruling Princes of India, Zemindars and others. No greater incentive for such an organization is needed than the example of the South Indian Mathematician above-mentioned.

But while we talk philosophically about such schemes, and indulge in these visions, we ourselves have a heavy responsibility on our own shoulders. We are in fact in the beginning of our existence and we have not as yet gone beyond the foundation of the edifice for the raising of which we started our Society; and though we may increase in numbers and have enough finances at our disposal, yet unless everyone of us works with heart and soul and contributes his individual share in one form or the other, our Society cannot attain that vigour of life so essential to its growth and welfare. In conclusion, therefore, the Committee appeal strongly to one and all who share in the proud privilege of being members of the Indian Mathematical Society, to realize their responsibilities and contribute their mite for the achievement of the laudable object for which this Society has been primarily founded.

Our Second Conference meets under favourable circumstances. The dire war has now nearly ended; the peace is in sight and humanity is again taking a breath of relief. Again, at the time of our First Conference held at Madras on the 27th, 28th, and 29th December 1916, the Bombay Government alone granted facilities to our members to attend the conference in the form of actual travelling allowance; but on this occasion, we feel greatly flattered to find that not only the Bombay Government, but the Governments of Madras, the Central Provinces,

and the Native States of Mysore, Hyderabad, Travancore, Baroda and Kolhapur, the Mission authorities at Beawar—all have granted facilities to our members to come to the conference. Our best thanks are due to all of them for their appreciation and recognition of our Society.

In conclusion, our Committee beg to thank most heartily one and all who have so cheerfully in one form or another assisted the Society ever since its foundation during all these years. It would be invidions to single out particular names. We appreciate thankfully their efforts and sincerely wish them to continue their support as hithertofore.

Presidential Address by Professor A. C. L. Wilkinson.

At the Fourth International Congress of Mathematicians held at Rome in 1908, a commission was appointed to make a comparative study of the Mathematical teaching in the Secondary Schools of different countries. The commission represented practically all the European countries, the United States, Japan, Australia, Canada, Cape Colony Mexico, Brazil. That India was not represented and took no part in the collecting of material was probably due to the fact that on 1908 no Indians had joined the conference. A preliminary report was made to the Fifth International Congress held at Cambridge in 1912, but its work was not concluded. Amongst several outstanding points left for consideration I may mention one, *viz.*, "the practical and theoretical training for teachers of Mathematics."

As a result of the enquiries instituted, we now have a vast body of literature dealing with the teaching of Mathematics in nearly all civilized countries, includiug not only the general teaching, but also special teaching as required for physicists, Engineers and other scientists. In 1912 it was stated that the reports published so far amounted to over 9000 Octavo pages; Germany leading the way with a report in 6 volumes, France following with a 5 volumes report, England, Japan, Belgium, 2 volumes each. It may be noticed that the reports are all in one of four languages, English, French, German, and Italian.

In directing your attention today to this subject of the teaching of Elementary Mathematics, I am guided not only by its paramount importance but also by the fact that it is one of the objects with which our Society was founded, but which up to now has been, I fear, neglected namely, the formation of branches for promoting the study of Elementary Mathematics and the improvement of Mathematical teaching.

Mathematical teaching naturally covers a wide field, commencing from the first steps of arithmetic of the Kindergarten Schools, we pass from the study of arithmetic in the primary school to the study of mathematics, that is Arithmetic, Algebra and Geometry in the secondary school, and then on to the more advanced study of these same subjects with Trigonometry, Analytical Geometry and even the elements of the Calculus added in the first stage of our University work until we arrive at the higher stages of Honours B.A. and M. A. teaching and lecturing.

I propose to confine myself to the study of Mathematics in the secondary schools, where great changes have recently taken place in regard to elementary textbooks, teaching and the syllabus for University examinations. In England these changes were largely brought about by the efforts of the Mathematical Association with its journal "The Mathematical Gazette." It is true that for some time attention was chiefly concentrated on the improvement of Geometrical teaching; and we passed from "Euclid edited" to "Euclid revised" and then to "Euclid abolished." In this respect France had anticipated us by over a century; the first edition ot Legendre's *Elements de Geometrie* appeared in 1794, and on this was based most of the textbooks in use on the Continent of Europe and United States; England in fact remaining the last stronghold of Euclid unadulterated. Reform in Geometry effected, it was natural that Arithmetic and Algebra should next be dealt with and the results of many of the changes that have recently been advocated in England are to be seen in the corresponding new types of textbooks and the changes in University curricula throughout India. At the time of the formation of our society, most of this work was accomplished, reform was generally accepted and was being actively introduced into our schools and Colleges. But has not the time now come for us to endeavour to weigh in the balance these reforms, to sum up the results of the changes so far effected, to compare our own methods with those of other countries and so determine whether further changes are not now indicated?

Let me commence by suggesting two fundamental questions for our immediate consideration.

(1) Why do we teach Mathematics as a compulsory subject, arithmetic in the sense the word is commonly employed, elementary algebra and a course in geometry corresponding to the first four or six books of Euclid? Granted that some or all of these subjects should form part of every

educated man's equipment, to what extent should each be taught compulsorily ?

(2) Is it necessary to maintain the ancient water-tight compartment system of three groups of arithmetic, algebra and geometry ?

It is a Mathematician's privilege to deny the obvious or axiomatic, to refuse to accept as necessary the most elementary principle without enquiry. The reducing of geometry to its fundamental hypotheses led to the development of non-euclidean geometry, a subject still very little understood outside the small circle of professional Mathematicians ; again the logical basis of algebra leads one to consider the abolishing of the most obvious properties of numbers, the associative and commutative, and develope new sciences. Therefore we may seriously claim the right to investigate the root basis of the presence of Mathematics in our present system of education, without respect of authority, until we find that authority justified by argument or result of experiment. I shall assume that no one desires to dispute the axiom that every citizen of a civilized country must learn the three R's, reading, writing and arithmetic ; language comes naturally, it is acquired as a child's mind develops almost unconsciously ; every savage speaks his own language and counts on his fingers or tallystick; and until quite recently, in Europe and Asia few men could read, write and cipher. But in these days the labour of acquiring these three arts is so insignificant in comparison with the lifelong advantages and pleasure to be obtained therefrom, that we even realise that it is a duty to enforce this minimum amount of education on every citizen. But having taught him to read, we do not insist on his deciphering Egyptian hieroglyphics or Assyrian Cuneiform ; having learnt to write, it remains optional whether he acquires proficiency in shorthand or the art of engraving ; and therefore, why having learnt arithmetic, enough that is for his practical needs, should he be compelled to manipulate complex fractional or decimal reductions or the thousand and one tricky problems that you will find in every well ordered textbook ?

The primary reason for teaching these subjects is, I take it, historical ; a secondary (and now the primary) reason is the value that we may put on the study of elementary mathematics for training and developing the reasoning powers. As far as Indian schools are concerned, the foundation of the Indian Universities about 1860 led to the introduction of a well defined school course on arithmetic, algebra up to quadratics, and the first three or four books of Euclid, who at that time still reigned

in England unchallenged. All this was of course based on the corresponding English school course and was what a boy of 15 or 16 might be supposed to know. It is therefore necessary to go back to the history of education in Europe in order to realise the gradual development and extension of the School and University curriculum. The basis of elementary education in medieval times was Latin, logic and rhetoric or metaphysics. This was a three years course, called the *trivium*, and led up to the B. A degree. This was followed by four years spent in the study of the *Quadrivium*, arithmetic, geometry, music and astronomy. But frequently, apart from making a declaration to the effect that he had attended lectures on the subjects, a man continued to study logic, metaphysics or theology. No tests of proficiency were imposed in the *quadrivium*, and we learn that at Oxford as late as 1450 A.D. a student seldom succeeded in mastering more than the first two books of Euclid. These were the days when all learned men wrote and spoke in Latin, when neither the Hindu number notation nor the Arabic science of Algebra had reached Europe. In these early days Arithmetic was a formidable science, based on the Roman system of enumeration, and its most difficult as well as essential purpose, the calculation of the date of Easter and the other moveable festivals of the Church that depended on Easter. The works of Boethius, 500 A. D, formed the basis of all that was written up to the middle of the thirteenth century when the Hindu notation became well established. The history of the modern textbook in England may be said to date back to 1540, when Rebort Recorde published "*The Grounde of Artes*", of whom De Morgan wrote that "His memory deserved a much larger portion of fame than it has met on several accounts. Hewas the first who wrote on Arithmetic and Geometry in English, he introduced Algebra into England and was in all probability the first Englishman who adopted the system of Copernicus."

Shortly after in 1585, decimals were invented by Stevinus of Bruges; logarithms by Napier in 1614, and in 1631 there appeared Oughtred's "Clavis Mathematica," originally written in Latin but translated into English as "The Key of Mathematics, new forged and filed". This was the culminating point in the historical development of the science of Arithmetic and Algebra as derived from the Arabs. We now approach the stage when Arithmetic was to become utterly trivial and mechanical when such a book as Cocker's Arithmetic, "being a plain and familiar method suitable to the meanest capacity for the full understanding of that incomparable art as it is now taught and practised by the ablest schoolmasters in town and country. Composed by Edward

Cocker, late practitioner in the Arts of Writing, Arithmetic and Engraving. Being that so long promised to the world.", was to receive such commendation, that "according to Cocker" became a byword for accuracy. This much over-rated work was published in 1700, and the next century and a half comprises numerous publications, now happily forgotten, in which Arithmetic was still further debased. Accuracy does not require endless summations of long columns of pounds, shillings and pence; lengthy compound practices, whose complexity depends on an utterly unscientific set of tables of measurement; incredibly distant G.C.M's, or unintelligibly large L. C. M's, terrifically complicated vulgar fractions which ultimately reduce to unity; the development of the intelligence can be attained without the introduction of compound proportions, true discounts, medial and alternate alligations. The signal of revolt was first hoisted by De Morgan, born at Madura in 1806, who pointed out that processes and not results, logical arguments and not tricks, were the scientific foundations on which arithmetic should be based, that mere dexterity of manipulation with an entire absence of reasoning and an entire absence of statement was valueless. What De Morgan wrote nearly a century ago is well worth reading even today and a new edition of his work "On the study of Mathematics" has been recently published in America by McCormack. The first sign of improvement was apparently in Colenso's Arithmetic, which had a considerable vogue in the middle of last century and on which all succeeding textbooks have been based. During the last ten years there has been a fresh revolt against the traditional teaching and scope of arithmetic and at the present moment the educational world is being overwhelmed with an avalanche of odd textbooks, English and Indian: some consisting of only graduated exercises leaving explanations of theory to the teacher, others full of typically worked out examples intended for home or private study; some with answers, some without; some especially intended for the University Matriculation, others for the working man or for business-men or evening continuation classes; some for kindergartens; others for standards 1 to 4; each and every one adorned with a preface pointing out its special applications, innovations or merits. Perhaps some of the most marvellous are those written for Indians. These are veritable mines of information; miniature textbooks of political economy and banking, explaining the intricacies of mercantile law under stocks, giving every conceivable table of measure, English and Indian, obsolete and current; your Indian student was equally prepared to distinguish between a hogshead of beer and a hogshead of wine, having we will hope never tasted either; he could purchase a hide of land, inform you of the ancient value of a tester or

moidore, was equally familiar with the dnodecimal process as with Shuvankar's, could draft a bill of exchange on a London Bank or cash a hundi at the local Marwaris.

Returning to my first question, you will perceive that the answer to that part of it relating to arithmetic is exceedingly complex. We teach certain portions of arithemetic for their utility in our daily life; we teach other portions, as logical exercises, the rest and, I fear, greatest portion is taught because it has gradually become incorporated in the textbooks and because it is presumed an examiner may set a question either relating to the foolish individual who turns all the taps of his bath on simultaneously, including the waste pipe, and times its filling with a stop watch, or the anxious secretary of a club who is puzzled over his handicap in a billiard tournament. It may sound strange, but there is a certain quaint humour about many of the arithmetical problems of the past and even present generations. I expect that most of you know the boatman's difficulties with the wolf, goat and basket of cabbages, and have heard of the Roman widow whose share of her lamented husband's estate depended on the sex of a posthumous child, which proved to be twins, a boy and a girl. It was however left for a Madras examiner to immortalise Aesop in Arithmetic in 1867 as follows :—

> A crow wishing to quench his thirst came to a vessel which contained 23 cubic inches of water. The crow being unable to reach the water picked up several small stones, each $\frac{1}{3}$ of a cubic inch in size and let them drop into the vessel until the water came to the top of the vessel. If the size of the vessel was such that it would exactly hold 73 cubic inches of water, find the number of stones dropped in by the crow.

I think that the most vividly realistic problem I have ever come across is the following from an Indian Textbook. Like the stories of Balbus in Lewis Carrol's Tangled Tales, it requires the art of the painter to really depict it.

> Two monkeys having robbed an orchard of 3 times as many plantains as guavas, are about to begin their feast when they espy the injured owner of the fruits stealthily approaching with a stick. They calculate that it will take him $2\frac{1}{4}$ minutes to reach them. One monkey who can eat 10 guavas per minute finishes them in $\frac{3}{8}$ of the time, and then helps the other to eat the plantains. They just finish in time. If the first monkey eats plantains twice as fast as guavas, how fast can the second monkey eat guavas?

The whole problem is so delicious in its sentiment and accidental misprint, that I feel that it calls for no comment—except that this side of arithmetic can well be abolished and the whole of this class of problems relegated to the comic history of arithmetic. Who are really responsible, the textbook writers or the examiners? The evils of examinations are well known, the remedy for them hard to find. A boy can be specially prepared or crammed for an examination and thereby obtain a post for which he is utterly unfitted. A man's intellectual life may be ruined by the process of accumulating different kinds of knowledge for reproduction in the examination room in a match against time and then allowing it to escape more quickly than it was poured in. The process of preparation may bring deleterious effects to the health or constitution of those undertaking it. It is not even an infallible process for obtaining the best or cleverest men. Too much importance may be attached to certain subjects on account of their intrinsic difficulties or the facility with which an examiner can floor the candidate. Each and every one of these evils are exemplified in arithmetic as crammed and sometimes examined in.

The methods of teaching arithmetic in the earliest stages is a matter for the pedagogist or psychologist, and the writings and lifework of Grube, Froebel, Pestalozzi and Madame Montessori and others too numerous to mention show how little our ancestors knew or cared about the psychology of the child's mind. We are concerned with the final stage, namely those arithmetical exercises in which examiners delight and the consequent abnormal growth of the arithmetical text-book. We have too, out in India, problems to solve that are essentially different from those in England or other countries and it is our first duty to organise enquiry into the teaching and examining of arithmetic; to obtain authoritative opinions from those engaged in school teaching and finally to study the teaching in other countries than England. I may perhaps just mention some very simple points of enquiry.

Should the multiplication table be learnt beyond 9×9?

Should not a simple table of weights and measures be drawn up?

In such case should not those measures only in common use in daily life be included?

Should compound practice be retained in our textbooks?

Should "true" discount be taught?

Should not more importance be attached to an algebraical solution of a problem than a so-called arithmetical one?

A second subject in the quadrivium was Geometry—still today a pivot in our system of secondary education. Euclid lived and wrote *The Elements* 2200 years ago at Alexandria. The medieval student who studied Boethius, learned the enunciations of all the propositions of the first book and those of a few selected propositions from the third and fourth books. Boethius only troubled to give proofs of the first three propositions. The whole of the *Elements* were first translated from Arab sources into Latin by Adelhard in 1120, and in the fifteenth century the first six books were generally studied in continental Universities for the Master's Degree. The first English translation was made by Billingsley and printed in 1570, followed by editions by Barrow in 1650, Whiston in 1702, Simson in 1756 and the multitude of textbooks of the 19th and 20th centuries.

While Euclid's work is of a very high order of accuracy and rigour and most valuable as a logical discipline, we can hardly suppose that it was ever intended for schoolboys, and even as a subject in the quadrivium, few men paid any attention. However last century an English schoolboy would start learning Euclid at the age of 11 or 12, and gradually work his way through the first six books. It was however not uncommon, I almost said universal, to evade the logical difficulties of the 5th Book, a real masterpiece of logical reasoning, and to pass on to the 6th Book after committing to memory a long-winded definition of proportion and a still more obscure definition of duplicate ratio. Gradually it began to be realised even in England, the last stronghold of Euclid, that modern progress in arithmetic and algebra demanded a different treatment of the science of Geometry. The labours of the Association for the improvement of Geometrical teaching are evident today in the revised syllabuses of all Indian Universities. All teachers have not welcomed the reforms, though there are probably few who would retain the pomp of the old time regime, when not a single symbol of abbreviation was permitted, much less any algebraical processes in combining rectangles and squares, when deviation from the written text was to be accounted a mistake. For some Euclid was an end in itself, for others but the threshold of a mathematical education. At first sight it would appear that Euclid is essentially based on sense perception, as for example those well known fallacies of the figure given in Rouse Ball, Mathematical Recreations; and most teachers, possibly unconsciously, make a considerable appeal to the sense perceptivity of their pupils by the carefully constructed diagrams used in demonstrating the proofs of the various problems and theorems. The visual sense can be assisted or even replaced by the tactile sense ; it is this in fact that now forms the basis of so-called practical Geometry.

Several mathematicians have continued their mathematical work after losing their eyesight, but I believe that there is only one instance on record of a man practically born blind (he lost his sight a few months after birth), who has achieved distinction in Mathematics. That was Nicholas Saunderson, born in 1682, who became Lucasian Professor of Mathematics at Cambridge and whose lectures on Algebra and fluxions were published after his death. Some recent experiments in teaching Geometry to the blind possess great interest and in fact a month's teaching of Geometry, or Arithmetic, in a blind school would, I believe, considerably benefit most teachers. Geometry as conceived by Euclid possesses many difficulties and we have added to those inherent difficulties by the invention of riders and problems of construction in such numbers that few textbook writers let a boy off with less than five hundred to a thousand. I do not wish to dogmatise, but speaking of my own boyhood 30 years ago, I believe that only a very small percentage of English schoolboys benefited by the logical reasoning of Euclid, and in India, apart from the same difficulties, a boy had to learn the subject in a foreign language, with well known consequences—a large percentage of Indian boys memorised the propositions and few masters ever found time to solve many riders in class. This evil was not altogether unknown in England, for Pollock, who was senior Wrangler in 1806, relates with pride that he could repeat the first book of Euclid word for word and letter for letter.

Let us be careful that (1) the new course does not become too stereotyped, (2) that it remains rational and reasonable. Are we to separate the sense perception from the abstract thought, or rather at what age shall we commence to combine the two? sI the process to be gradual or shall we completely separate the two aspects?

Where shall we stop? Must every educated man know the property of the nine-point circle? Is there not more conviction that the areas of triangles are as the squares of their sides to be found in the simple illustrations of bisecting, trisecting the sides or in building up $1+3+5+7$ rectangles, than by a formal proof, whether algebraical as at present taught, or Euclidean as in the past when "Similar triangles are in the duplicate ratio of their homologus sides" proved a sure net for the examiner to entrap the unwary. The discovery of incommensurable magnitudes was of supreme importance in the development of Greek Geometry, but it is of no importance to the practical engineer. If we seriously thought that an exact study of magnitnde was essential for the development of the culture of the ordinary individual, then let us insist

on the very careful reading and appreciation of the 5th and 10th books of Euclid or their equivalent.

And there is still a third subject, Algebra, a very recent human invention, not included in the quadrivium. Dating back to the Arab writer Abu Ja'afar Mohammed (about 820 A.D.), known as Al Khowarazmi, we can recognise three distinct periods in its development. The rhetorical, when the equations were written in words; the syncopated when the words are abbreviated as in Recorde and Cardan; the modern ort symbolic algebra of Vieta, Descartes—which exists today. The teaching of this subject has been more vitiated by the examination evil than any other. Its ogic was wholly neglected, it existed only as a collection of rules whose object was the solution of examination questions. Here again during the last ten years a great revolt bas taken place and the methods both of teaching algebra and the arrangement of the text books have undergone great and beneficial alterations. But tradition still remains firm, chapters on ratio, proportion and even variation still disfigure our textbooks, the study of the arithmetical and geometrical series and their natural corollary, logarithms, is postponed until elaborate chapters on surds and indices have been mastered and an elaborate study of the harmonical progression still wearies most schoolboys.

Considering first the standpoint of the usefulness of elementary mathematics to other sciences—engineering, physics—much of the time spent on algebra must be regarded as entirely wasted. Is the process of H.C F ever employed except in pure mathematics? Taking the second standpoint, the value of mathematics in developing the mind, the carefully arranged sets of standard examples must dull any mind, and so far from developing the reasoning powers, by the time a student comes to permutations and combinations, having depended in the mejority of cases, entirely on an uncomprehended set of rules for his past exercises, he fails lamentably on the first occasion that some real mental reflection is required from him. Our course in algebra is too long because we dare omit nothing: generations of text book writers have been in keen competition with generations of examiners, the one to leave nothing out of his book that could possibly be set, the other to set something not to be found in the book, and the result the despairing candidate who religiously works every example in his textbook and any other he can get hold of.

I hope that I have succeeded in making it clear, by this brief resume of the history of the three school subjects in mathematics, that what we

are teaching today to schoolboys from 12 to 17is the result of the wisdom and research of many generations. That it is possible for the average boy to assimilate this accumulated knowledge of centuries in such a brief period is possibly one of the best reasons that we can give for the study of mathematics up to a certain point. The irrefutable logic of Geometry, the perfection of the Symbolism of Algebra, are inheritances from the past and it is not possible to calculate, though we might estimate from statistical experiments in which the present rising generation would unfortunately have to play the role of the corpus vile. the effect on the development of the child's mind of three or four years' study of mathematics. We must to some extent accept an unanimous verdict of all civilized countries, but when we enquire to what extent the study of compulsory mathematics should be carried, we shall find very considerable differences both in theory and practice. A study of the school text-books of America, France, Germany and Italy will show you how great is this divergence in practice and will, I think, lead you to enquire whether we are not even today requiring too high a standard from our youth. It was the accident of circumstance that led India to adopt the system prevalent in England in 1860 and to change in conformity with changes that took place subsequently in England. In the past, India's contribution to elementary textbooks has been rather in the nature of creating storehouses of problems by the solution of which in ever increasing numbers, no boy should fail to achieve success in the all-important matriculation examination that governed his future prospects in life. How utterly vicious a system had developed we all admit, and the present reorganisation of elementary mathematical education that has been going on for the last two decades requires the closest co-operation of all concerned, the expert and the practical teacher can each learn their deficiencies only from the other.

I need only deal very briefly with the second fundamental question propounded, namely, the complete separation of the three subjects now taught as Arithmetic, Algebra and Geometry.

In many other countries, notably Germany, arithmetic and algebra are not separated and the system that provides a mathematical work for each standard or even for two standards possesses many advantages for the student if it fails to give much freedom of choice to the teacher. The complete treatise on arithmetic that includes the teaching of the addition and multiplication tables of the kindergarten schools with the stocks and problems of the secondary school appears to me a complete anomaly. Many writers have likened the teaching of elementary mathematics to a spiral or a circle continually enlarging; multiplication,

division, fractions, Rule of three should not be completely done at one stage. The simpler portions can be done first, fractions may be added long before they are divided, decimals may be used in practical measurement long before they are multiplied : the simple equation in algebra may with advantage precede the mechanical operations of the division and multiplication of polynomials, and no distinction need be allowed to remain in a boy's mind between a problem in arithmetic and one in algebra.

Enquiry into the teaching and extent of compulsory mathematics is by no means finished. Tradition still prevents many changes from being introduced into England, a country in which the cult of Euclid is not dead even today. Nor shall we answer these fundamental questions without collecting the opinions and experiences of a large body of teachers, without studying the system in vogue in other countries and lastly without studying the theoretical writings of the pedagogists. Therefore I urge upon you the necessity of carrying into practice that article of our Constitution that provides for the formation of branches in the various presidencies, whose duty it will be to promote the study of elementary mathematics by setting on foot enquiries into the methods of teaching, the extent to which each subject should be taught and numerous other questions which will be suggested; with the materials thus collected classified and discussed, we shall be in a position to advocate such further reforms as may be considered necessary and thus in the future be able to initiate where in the past we have only followed.

Introductory Notes to Differential Geometry.

By Eric H. Neville.

Introductory.

It is a commonplace of writers on differential geometry that the arc element of a curve is essentially positive, and there is universal agreement that the curvature of a curve in space also is positive*. That these two conventions are sometimes troublesome nobody† will deny; the object of the following paragraphs is to shew that they are neither necessary nor desirable even in the earliest stages, and that in fact the subject can be developed smoothly and rapidly without either of them.

* See for example Bell, pp. 277, 283, 288, Eisenhart, pp. 10, 14, 181, Forsyth pp. 2, 4. Continental authors make the same assumptions.

† The difficulties are faced scrupulously by Eisenhart, as for example on pp. 118, 182. – There are of course writers careless enough to break without suspicion conventions by which they profess to be bound.

The essence of the treatment I adopt is in the definition of a curve. The usual view is that a curve *acquires* the order of its points from the parameter by means of which it is studied. I assert that on the contrary *the curve of kinematical geometry is not the mere aggregate of its points, but is this collection of points with a definite serial order that has nothing to do with any particular variable used in discussing the curve.* Of any two points of a kinematical curve one *precedes* the other.

If P, Q are two distinct points of a curve, the secant PQ is not the Euclidean line through P and Q but the ray or directed line on which P and Q have the same order as on the curve: if P precedes Q on the curve, the secant PQ is the ray through P and Q whose direction is from P to Q, and the secant QP is this same ray. The tangent at a point O also is a ray, the limit of the secant PQ as P and Q approach O independently.

The chord PQ has its length measured along the secant; the length of PQ is therefore positive or negative according as P precedes or follows Q, and the length of QP is the negative of the length of PQ.

Analysis begins when the positions of a current point O on the curve are correlated with the values of a real variable t. If Q lies between P and R, then t_q lies between t_p and t_r; hence for a given correlation the sign of $t_q - t_p$ depends only on whether P precedes or follows Q. The sign of $t_q - t_p$ when P precedes Q is called the sign of the correlation, and if this sign is positive t is said to be an increasing or a positive variable for the curve.

The length of the chord PQ divided by $t_q - t_p$ is a definite number with the sign of the correlation; on a regular curve this number tends to a unique limit as P and Q tend in any manner to O, and this limit, which will be denoted by w, is the speed of O with respect to t. If T is a second variable for the same curve and w is the speed with respect to T

$$\mathrm{W} = \lim \{ \mathrm{PQ}/(\mathrm{T}_q - \mathrm{T}_p) \} = \lim \{ \mathrm{PQ}/(t_q - t_p) \} \{ (t_q - t_p)/(\mathrm{T}_q - \mathrm{T}_p) \}$$
$$= w\, dt/d\mathrm{T},$$

and therefore

$$\int_{\mathrm{T}_p}^{\mathrm{T}_q} \mathrm{W}\, d\mathrm{T} = \int_{t_p}^{t_q} w\, dt;$$

the value of the integral, which is thus shewn to be independent of the particular variable, defines the arc of the curve from P to Q. Since w has the sign of the correlation, the arc from P to Q is positive or negative according as P precedes or follows Q. The arc to the current

point O from some definite point is denoted by s; the speed w is the derivative ds/dt, and since the arc is always found, in practice as well as in theory, by integration of the speed, w is sometimes called the arc function of the curve for the variable t. By the adoption of the arc for variable, the speed can be made unity.

If OT is the tangent at O to a curve Γ and Ω is a fixed point, the point $\mathcal{T}$ at unit distance from Ω in the direction OT traces the tangential indicatrix of Γ. It is assumed that this indicatrix has no singularities and is itself an ordered curve, but the order of the points on the indicatrix is not made dependent on the order of the points on the original curve: if P precedes Q on Γ, the tangential image of P may either precede or follow the tangential image of Q on the indicatrix.

Since Ω is a fixed point, the motion of $\Omega\mathcal{T}$ takes place in the plane through Ω containing the tangent to the curve described by $\mathcal{T}$, and since O moves in the direction OT (possibly with a negative speed), the motion of OT is in the parallel plane through O. This plane through O, with a cyclic direction yet to be assigned, is the osculating plane, and the principal normal, a normal in this plane, is therefore parallel to the tangent to the indicatrix. Hence it is possible to define the direction of the principal normal to the original curve and the direction of the tangent to the tangential indicatrix to be the same; in effect, sometimes the order of the points on the indicatrix is known and determines the standard direction OP of the principal normal, but often it is the latter direction which is known and serves to distinguish the standard direction of the tangent to the indicatrix and in this way to determine the order of precedence on the indicatrix.

The ratio of the speed of $\mathcal{T}$ to the spead of O is independent of the variable t and measures the curvature of the path of O; in other words, the curvature, which is denoted by κ, is the speed of the tangential image of O with respect to the arc of the curve described by O. The sign of κ depends on the relation between order along the original curve and order along the indicatrix, κ being positive if precedence corresponds to precedence. If no other circumstances influence the choice, it is natural to give the points on the indicatrix the order which makes κ positive, but if the direction of the principal normal is otherwise decided the sign of κ is both determinate and significant.

The course taken in plane geometry is to define the standard direction of the normal as making a *positive* right angle with that of the tangent, when the sign of the curvature determines on which side of the tangent the curve is to be found. Quite analogous is the case of a geodesic on any bifocial surface: the principal

normal of a geodesic must lie along the normal to the surface, and if to the former of these rays is assigned the direction of the latter, it becomes possible to express the curvature in direction and amount in the simple assertion that the curvature of the curve is the normal curvature of the surface. Another example may be drawn from the discussion of Bertrand curves—pairs of curves with a common principal normal; strictly speaking, the assumption that curvature must be positive renders the investigation usually given purely tentative, and the work would be misleading if the assumption were anything but gratuitous.

For kinematical geometry the osculating plane is the plane OTP with the cyclic direction determined by the convention that there is a positive right angle from the tangent OT to the principal normal OP. The spatial convention distinguishes one of the two directions at right angles to this plane as the positive normal direction, and it is this which is the direction of the binormal OB; in other words OB is at right angles to both OT and OP and there is a positive right angle from OT to OP round OB. The binormal indicatrix is traced by the point B at unit distance from Ω in the direction OB.

The torsion of the curve measures the spin of the plane OTP or the velocity of the point β with respect to the arc. Although the arguments for supposing torsion essentially positive are neither weaker nor stronger than in the case of curvature, it has long been recognised that torsion has a significant sign, the reason for the perception being that the direction in which torsion is to be measured can be assigned by a single universal convention. If the spin of the plane OTP were not wholly about OT, the tangential image τ would have a velocity at right angles to this plane in addition to its velocity in the direction OP, and this would contradict the definition of the principal normal. Hence it is always about the tangent that the osculating plane is spinning, and always parallel to the principal normal that the binormal image is moving; the torsion can therefore be given a definite sign either by the choice of OT rather than its reverse as the direction round which to measure the spin of OTP, or by the choice of OP rather than its reveres as the direction in which to measure the velocity of β. The two plans give opposite signs to the torsion, but unfortunately both have been used, so that care is necessary in the comparison of formulae from different writers; the first convention is now accepted as the better, and it is the spin of OTP about OT which, denoted by σ, is called the torsion. But to adopt either convention in preference to postulating that torsion must be positive is to admit that the current point of a binormal indicatrix must sometimes be held to have a negative speed.

SHORT NOTES.

On the factorisation of two large numbers.

I have succeeded in factorising two of the large numbers that appear in the second column of the table prepared by William Shanks (on the determination of a prime number, the number of figures in the period of the reciprocal being given) and published in the *Proceedings of the Royal Society of London*, (June 18. 1874), Vol, XXII, p. 383. They are 165573604901641 & 99999 99999 00000 00000 99999 99999 00000 00001 appearing against 60 & 100 in the table referred to above. The former of these is equal to 4188901×39526741, and the factors of the latter are 99004 98006 98004 9901 and 10100 50200 70200 50100 1.

I am not aware of these factors being given previously. With the former of these results the factorization of $10^{60}-1$ becomes complete. In fact it is equal to

$$3^3.\ 7.\ 11.\ 13.\ 31.\ 37.\ 41.\ 61.\ 101.\ 211.\ 241.\ 271.\ 2161.$$
$$3541.\ 9091.\ 9901.\ 27961.\ 2906161.\ 4188901.\ 39526741;$$

all primes.

The factorisation of $10^{100}-1$ now stands as follows

$$3^2.\ 11.\ 41.\ 101.\ 251.\ 271.\ 3541.\ 5051.\ 9091.\ 21401.\ 25601.$$
$$27961.\ 182521213001.\ 78875943472201.$$
$$9900498006980049901.\ 1010050200702005010 01;$$

all but the last four are known to be primes.

ALLAHABAD, *7th January*, 1919. N. B. MITRA.

On Questions 942 & 943.

Mr. Leather's solution of Question 943, (p. 448 Vol. X) may possibly be taken to imply that that problem was taken from the Punjab University M. A. Papers of 1918. We therefore wish to state here that this problem and the special case published as Question 942 were composed by Mr. Hem Raj who had arrived at the result independently. He made enquiries as to whether this property of the co-efficients was already known or not, but as it was new to every one to whom it was communicated, the questions were accepted for publication. It now appears that, by a coincidence, one of the questions was set at the Punjab M. A. examination last year. Professor Wilkinson has since traced the result in Question 943 so far back as 1899 when it was published by Glaisher in the *Quarterly Journal of Pure and Applied Mathematics* (Vol. 31). The special case in Question 942 appears to be due to Wolstenholme. Dr. Glaisher's proof is summarized by Professor Wilkinson as follows :—

A_r = the sum of the products of $[1, 2, 3 \ldots n-1]$ r at a time; denote any term of the product by $p.q \ldots u(n-u)v(n-v)\ldots$ where the conjugate factor $n-p$, $n-q \ldots$ do not occur.

In A_{2r+1} we have sets of terms

$$p.u(n-u)v(n-v)\ldots$$
$$pqr\ u(n-u)v(n-v)\ldots$$
$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

The first set of terms, by associating each term with its complement $(n-p)u(n-u)\ldots$

$$=n\frac{(n-2r-1)}{2}\Sigma_{2r}u(n-u)v(n-v)\ldots.$$

The second set of terms gives rise to an element

$$\{pqr+(n-p)q.r+p(n-q)r+pq(n-r)+(n-p)(n-q)\}$$
$$+p(n-q)(n-r)+(n-p)q(n-r)+(n-p)(n-q)(n-r)\}\,u(n-u)v(n-v)\ldots$$
$$=n^3u(n-u)v(n-v)\ldots$$

Thus the second group of terms is

$$\left(\frac{n-1}{2}-r+1\right)C_3n^3\Sigma_{3(r-3)}\{u(n-u)v(n-v)\ldots\}$$

Hence $A_{2r+1}=\left(\frac{n-1}{2}-r\right)C_1.n\ \Sigma_{2r}u(n-u)v(n-v)\ldots+$

$$\left(\frac{n-1}{2}-r+1\right)C_2n^2\ \Sigma_{2r-2}u(n-u)$$
$$+\left(\frac{n-1}{2}-r+2\right)C_3n^3\Sigma_{(2r-4)}\,u(n-u)\ldots+\ldots$$

Similarly

$$A_{2r}=\Sigma_{(2r)}u(n-u)v(n-v)\ldots+\left(\frac{n-1}{2}-r+1\right)C_2.n^2\Sigma_{(2r-2)}u(n-u)\ldots$$
$$+\left(\frac{n-1}{2}-r+2\right)C_4n^4.\Sigma_{(2r-4)}u(n-u)\ldots$$

where the suffix to the Σ denotes the number of elements u, $n-u$, v, $n-v$,...occurring.

Now it is well known that A_1, $A_3,\ldots A_{n-2}$, $A_{n-1}+1$ are divisible by n when n is a prime, hence $\Sigma_{2r}\,u(n-u)v(n-v)\ldots$is divisible by n.

Thus A_{2r+1} is divisible by n^2 when n is a prime except for $r=0$. Incidentally, we have established that the sum of the products of 1^2, $2^2,3^2,\ldots m^2$ taken r at a time is divisible by $2m+1$,when $2m+1$ is a prime—a theorem due to Ferrers and proved as above by Glaisher, *loc cit*, §237.

EDITOR.

Astronomical Notes.

The Planets.

Most of the planets are favourably situated for observation at this time. Their positions on March 1, are given below :—

Planet	R.A.		Decl.			Constellation
Mercury ...	23^h	6^m	... 7°	$17'$	S	... Aquarius
Venus ...	0	13	... 0	14	N	... Pisces
Mars ...	23	46	... 2	19	S	... Pisces
Jupiter ...	6	25	... 23	29	N	... Gemini
Saturn ...	9	46	... 15	3	N	... Leo
Uranus ...	22	3	... 12	41	S	... Aquarius
Neptune ...	8	38	... 18	25	N	... Cancer

The Sun's position on the same date is R.A. 22*h* 46*m* & Decl 7° 52′ S ; so that, Mercury, Venus and Mars are evening stars. Saturn was in opposition on Feb. 14th.

Comets.

A new comet was discovered (probably the last for the year 1918) by Dr. Schorr at Hamburg (Bergdorf) on Nov. 23, 1918. Its positiou on Jan. 30 was R. A. 4*h* 9*m*, Dec. 16° N. The comet is too faint to be seen by telescopes of small aperture.

Mr. Denning gives the following list of periodical comets which are due to return to Perihelion in 1919 :—

Name	Period in year	Last observed return to Perihelion.
Brooks (1886)	5·594	1886 June 6
Finlay (1886)	6·540	1906 Sep. 8
Kopff (1906)	6.581	1906 May 2
Holnes (1892)	6.857	1906 March 14
Schaumasse (1911)	8·071	1911 Nov. 30
De Vico (1846)	75·25	1846 March 6.

De Vico's comet is interesting as being one of the six forming Neptune's family of comets. It will make its appearance in the southern sky.

Nova Aquilae.

It appears from a communication to the R. A. S., that one of the earliest observations of this Nova, after its outburst, was made by Mr. G. N. Bower of the Palms, Nungambaukam, Madras, who observed the Nova between 9 and 10 P. M. (Madras Time) on June 8, 1918. The earliest record is on a photograph taken at Harvard on 1888 May 22. Its photographic magnitude was then 10. 5. It has been found on 405 subsequent photographs which sufficiently indicate small fluctuations in the brightness.

A New Observatory.

The Smithsonian Institution U. S. A. has recently established an Observatory at Calama, Chile (lat. 22° 28′ S, long. 68° 56′ W, altitude 7300 ft.) for observing the solar variability for a period of years. The site is believed to be the most cloudless in the world.

T. P. BHASKARA SASTRI.

Nizamiah Observatory,
Hyderabad (Deccan).

SOLUTIONS.

Question 958.

(MARTYN M. THOMAS, M. A.) :—Establish the formula

$$\frac{d^n u}{dx^n} = x^{r-n}\frac{d^r}{dx^r}\left[x^q \frac{d^{q-r}}{dx^{q-r}}\left\{ x^{p-r}\frac{d^{p-q}}{dx^{p-q}}\left(x^{n-q}\frac{d^{n-p}}{dx^{n-p}}\frac{u}{x^p}\right)\right\}\right]$$

where $n > p > q > r$.

Solution by Sadanand, T. K. Reddi, and L. S. Vaidyanathan.

Let D stand for $x\frac{d}{dx}$ and Θ for the usual auxiliary operator.

We have

$u_1 = ux^{-p}$

$u_2 = x^{p-q}D^{n-p}u_1 = x^{p-q}\Theta(\Theta-1)\ldots\ldots\ldots(\Theta-n+p+1)\,x^{-p}u$

$= x^{-q}(\Theta-p)(\Theta-p-1)\ldots\ldots\ldots\ldots\ldots(\Theta-n+1)u$

$u_3 = x^{q-r}D^{p-q}u_2 = x^{q-r}\Theta(\Theta-1)\ldots\ldots\ldots(\Theta-p+q+1)u_2$

$= x^{-r}(\Theta-q)(\Theta-q-1)\ldots\ldots\ldots\ldots\ldots(\Theta-n+1)u$

$u_4 = x^r D^{q-r}u_3 = x^r\Theta(\Theta-1)\ldots\ldots\ldots\ldots\ldots(\Theta-q+r+1)u_3$

$= (\Theta-r)(\Theta-r-1)\ldots\ldots\ldots\ldots\ldots\ldots(\Theta-n+1)u$

$u_5 = x^{-n}D^r u_4 = \Theta(\Theta-1)\ldots\ldots\ldots\ldots\ldots\ldots(\Theta-r+1)u_4$

$= x^{-n}\Theta(\Theta-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\Theta-n+1)u$

$= x^{-n}D^n(u) = x^{-n}.\ x^n\frac{d^n u}{dx^n} = \frac{d^n u}{dx^n}$

Hence the result.

Question 959.

(M. T. NARANIENGAR) :—The sides AB, AC of a triangle ABC are produced to D, E, such that $BD = CE = BC$. If the circle ADE cut the straight line AII_1 in J, prove that $2I_1 J = II_1$

Solution by K. B. Madhava, R. D. Karve and F. H. V. Gulasekharan.

Joining D and E with J it is easily seen that DEJ is an isosceles triangle, and $DE = 2\,DJ\cos\frac{A}{2}$.

Again in the cyclic quadrilateral AEJD, we have

$$AJ.\,DE = (AE + AD)\,JE$$

which gives $AJ\cos\frac{A}{2} = a + b/2 + c/2$.

Also, $AI_1\cos\frac{A}{2} = s$, and $II_1\cos\frac{A}{2} = a$.

$\therefore$ $JI_1\cos\frac{A}{2} = \frac{a}{2}$, which gives the the theorem, $J = II_1$.

Remarks by K. J. Sanjana M. A.

More generally, if $BD = la$, $CE = ma$, denoting by a' and R' the side DE and the circumradius of the $\triangle$ ADE, we get

$$DJ = EJ = 2R' \sin \frac{A}{2}, \quad DE = a' = 2R' \sin A.$$

Hence by Ptolemy's Theorem,

$$JA.\ 2R' \sin A = 2R' \sin \frac{A}{2} (c + la + b + ma)$$

$$\therefore \quad JA = \frac{1}{2 \cos \frac{A}{2}} \{ 2R(\sin B + \sin C) + 2(l+m) R \sin A \}$$

$$= R \{ \sin B + \sin C + l(+m) \sin A \} / \cos \frac{A}{2}.$$

$$\text{Also } AI_1 = r_1 \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2};$$

$$\therefore \quad I_1 J = JA - AI_1$$

$$= \frac{R}{\cos \frac{A}{2}} \left\{ \sin B + \sin C + (l+m) \sin A - 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \right\}$$

$$= \frac{R}{\cos \frac{A}{2}} \{ \sin B + \sin C + (l+m) \sin A - \sin A - \sin B - \sin C \}$$

$$= R\,(l+m-1) \sin A / \cos \tfrac{1}{2} A = 2R \sin \tfrac{1}{2} A\ (l+m-1).$$

But $II_1 = 4R \sin \frac{1}{2} A$; therefore $\frac{I_1 J}{II_1} = \frac{1}{2} (l+m-1)$.

When $l = m = 1$, the result in the question is obtained.

If $l+m =$ a constant, p, whatever numbers l and m may be, J is a fixed point on AII_1, given by $I_1 J = \frac{1}{2} (p-1) II_1$; thus the circles like ADE are coaxal and all intersect in the two real points A and J.

If the ⊙ ADE cuts the external bisector $I_2 AI_3$ in K, we can similarly prove $I_2 K = 2R \cos \frac{1}{2} A\ (l-m+1)$; also $I_2 I_3 = 4R \cos \frac{1}{2} A$;

thus $$\frac{I_2 K}{I_1 I_2} = \tfrac{1}{2}(l-m+1) \text{ and } \frac{I_3 K}{I_2 I_3} = \tfrac{1}{2} (m-l+1).$$

Hence, if $l - m =$ a constant, q, the circles like ADE are coaxal and all intersect in A and the fixed point K, given by $I_2 K = \frac{1}{2} (1 \pm q) I_2 I_3$.

Additional solutions by N. P. Pandya and M. K. Kewalramani, N. B. Mitra, K. J. Sanjana and R. D. Karve.

QUESTIONS FOR SOLUTIONS.

1015. (N. SANKARA AIYAR, M.A.):—Show that $b^2-y^2=z^3-4p^2y^3=x^3$ has no solutions in non-zero positive integers unless p is of the form $(12\,m+4\pm3\frac{1}{2}\pm\frac{1}{2})$.

1016. (P. SAMUELS LALL) :—Can Rolle's theorem be proved if the condition that the function $f(x)$ is continuous in the interval in which the function is defined, be removed and in its place it be inserted that $f'(x)$ should exist at every point of the interval except perhaps at the ends ?

1017. (N. SANKARA AIYAR) :—The conic which touches the sides at their intersections with the external bisectors is drawn. Show that the fourth common tangent of this and the Steiner ellipse divides the sides in the ratio $a\,(b-c) : b\,(a-c)$, $b\,(c-a) : c\,(b-a)$ and $c\,(a-b) : a\,(c-b)$

1018. (SADANAND):—Prove that between certain limits

$$(1)\quad \frac{e^3-1}{e^2+1}\frac{\sin x}{1^3}+\frac{e^6-1}{e^6+1}\frac{\sin 3x}{3^3}+\frac{e^{10}-1}{e^{10}+1}\frac{\sin 5x}{5^3}+\ldots=0$$

$$(2)\quad \frac{e^2-1}{e^2+1}\frac{\cos x}{1^3}+\frac{e^6-1}{e^6+1}\frac{\cos 3x}{3^3}+\frac{e^{10}-1}{e^{10}+1}\frac{\cos 5x}{5^3}+\ldots=\frac{\pi^2}{8}$$

1019. (HEMRAJ) :—Show that if $(p+4)$ be a prime number and the B's stand, as usual, for the Bernoullian numbers,

$$B_1-B_3+B_5-B_7+\ldots+(-1)^{\frac{p-1}{2}}B_p=\frac{m}{n}-\frac{3}{2},$$

where $|\,m\,|$ and n are integers prime to each other and $|\,m\,|$ is a multiple of $(p+4)$.

1020. (ANANTARAMAN):—Construct a magic square of 81 cells with the following numbers at the places assigned :--

1st row :—5th cell, 25 ;
2nd row:—4th cell, 79 ; 5th cell 29 ; 6th cell 27.
3rd row:—3rd cell, 43 ; 5th cell 81 ; 7th cell 5.
4th row:—2nd cell, 7 ; 5th cell 13 ; 8th cell 19.
5th row (in full) :—61, 47, 9, 67, 41, 15, 73, 35, 21.
6th row :—2nd cell 63 ; 5th cell 69 ; 8th cell 75.
7th row :—3rd cell 77 ; 5th cell 1 ; 7th cell 39.
8th row :—4th cell 55 ; 5th cell 53 ; 6th cell 3.
9th row :— 5th cell 57.

1021. (HEMRAJ) :—A′, B′, C′, D′ are the orthocentres of the triangles B C D, C D A, etc., of a quadrilateral A B C D. Show that $\triangle$ A B C $= \triangle$ A′B′C′; but if ABCD is cyclic these two triangles are similar. Show also that AA′, BB′, CC′ and DD′ bisect each other.

1022. (M. K. KEWALRAMANI) :—Solve the following 16 systems of four equations each :—

$$\pm\ ax^2+bxy+cxz+dxw=p_a,$$
$$\pm\ by^2+cyz+dyw+ayx=p_b,$$
$$\pm\ cz^2+dzw+azx+bzy=p_c,$$
$$\pm\ dw^2+awx+bwy+cwz=p_d.$$

(Suggested by Q. 153.)

1023. (N. P. PANDYA) :—Find the minimum sphere which would contain all the following :—(1) a sphere of radius 1 in. (2) a square based prism length $1\frac{1}{2}$ in. breadth 1 in (3) a cylinder of height 2 in. on an elliptic base of axes $2\frac{1}{4}$ in. and $1\frac{1}{4}$ and (4) a regular tetrahedron of edge $1\frac{1}{2}$ in.

1024. (N. P. PANDYA) :—Find the condition that the maximum inscribed ellipse of a triangle may have its minor axis along the inFeurbach radius. Is there any scalene triangle of this description?

1025. S. R. RANGANATHAN :—Show that the determinant of $2n$ rows and columns :—

$$\begin{vmatrix}
\frac{1}{1!} & -\frac{1}{2!\,1!} & \frac{1}{3!\,2!} & -\frac{1}{4!\,3!} & \frac{1}{5!\,4!} & \cdots \\
0 & \frac{1}{1!} & -\frac{1}{2!\,1!} & \frac{1}{3!\,2!} & -\frac{1}{4!\,3!} & \cdots \\
0 & 0 & \frac{1}{1!} & -\frac{1}{2!\,1!} & \frac{1}{3!\,2!} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & 0 & \frac{1}{2!} & -\frac{1}{3!\,1!} & \frac{1}{4!\,2!} & \cdots \\
0 & \frac{1}{2!} & \frac{1}{3!\,1!} & \frac{1}{4!\,2!} & -\frac{1}{5!\,3!} & \cdots \\
1 & -\frac{1}{3!\,1!} & \frac{1}{4!\,2!} & \frac{1}{5!\,3!} & \frac{1}{6!\,4!} & \cdots
\end{vmatrix}$$

$$= \{\, 1^{2n+1}.2^{2n}.3^{2n-1}\ldots(2n)^2.(2n+1)\,\}^{-1}$$

1026. (S. R. Ranganathan) :—The source being a luminous point, show that in any plane passing through the point, the curves of equal illumination are either systems of lemniscates, or circles which envelop the lemniscates of these systems.

THE JOURNAL

OF THE

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Vol. XI.] APRIL 1919. No. 2.

PROGRESS REPORT.

1. *The Founder of Our Society.*—The honor of being the Founder of our Society belongs to Mr. V. Ramaswami Aiyar (whose photo accompanys the present number of the Journal), now Special Deputy Collector of Ramnad. His first letter proposing the formation of the Society, dated 25th December 1906, is contained in his letter, dated 3rd April 1907, announcing the formation of the Society under the name '*The Analytic Club.*' The latter was published in the leading Madras dailies of the 4th April 1907; and it is but fitting that we place it on record in the pages of our Journal. It will be observed from the dates, that our Society has just completed the twelfth year of its existence. The discovery of Mr. S. Ramanujan, B. A., F. R. S., one of the greatest living Mathematicians of the world, marks this period of its life. The founder of the Society devoutly hopes that the objects with which the Society was formed will be fullfilled to a greater degree in the future periods of its existence, and expects that all members will unite their best endeavours in the realization of this hope. The letter announcing the formation of the Society is printed below.

2. *Mr. S. Ramanujan, F. R. S.* This distinguished Mathematician has returned to Madras in somewhat indifferent health, after a prolonged stay at Cambridge. By his unique mathematical talents and by the amount of useful and original work, he has raised India in the estimation of the outside world. We extend our most cordial welcome to him and most fervently pray that he may be soon restored to his full vigour to prosecute his glorious work in the scientific world. The committee hope to show their mark of appreciation further by a suitable announcement in the next issue of the Journal.

3. *Mr. T. J. Mirchandani B. A., B. Sc.*—Demonstrator, D. J. S. College, Karachi, has been elected a member of our Society.

POONA,
1st April 1919.

D. D. KAPADIA,
Hon. Joint Secretary.

MR. V. RAMASWAMIAIYER'S LETTER, DATED 3RD APRIL 1907.

SIR,—I recently sent a proposal to some gentlemen interested in mathematics suggesting the formation of a small Mathematical Society. The proposal ran as follows :—

SIR,—I believe several friends interested in Mathematics have felt the present lack of facilities for seeing mathematical periodicals and books. This is a very great disadvantage we are suffering.

I propose therefore that a few friends may at once join and form a small Mathematical Society and subscribe for all the important Mathematical periodicals and, as far as possible, for all important books in Higher Mathematics.

We may call the Society "The Analytic Club" for the present, and have it in view to give it a broader basis with a suitable name by and by.

Our work immediately will be to obtain all the important periodicals and new books and circulate them to members. I shall be glad to undertake the duties of Secretary for the first year and do my best to promote the object in view.

If half a dozen members can be counted upon to join immediately and each subscribe Rs. 25 per annum, we shall be able to make a good start. The Annual Subscription may perhaps be somewhat less, say, if a dozen members can be had ; but even a dozen members paying Rs. 25 per annum would not suffice to enable the Club to obtain the more important books appearing every year. I propose therefore that the subscription be Rs. 25 per annum.

It appears to me necessary also that members should be prepared for a further sacrifice, and I propose that each member should send the journals and books he receives on to the next, and the last to the Secretary, at his own cost. This in effect would be to add about Rs. 5 more to one's subscription. I hope friends interested in mathematics will not consider this a too heavy sacrifice—at any rate initially, in giving the Club a start.

Will you kindly write to me if you are in favour of the proposal ? In case you are, I request you will send me your subscription of Rs. 25 for 1907, as early as possible, so that we may make a start at once.

This is only a tentative scheme and we may try it for a year and then introduce necessary changes.

I propose to consider the Club formed as soon as three friends have agreed to the proposal, making with me four members. Thereafter all business requiring determination by the Club can be done by circulation. Requesting the favour of an early reply. I remain, Sir, yours truly,

(Signed) V. RAMASWAMI AIYAR.

2. In response to this proposal (which I was able to send only to a very limited number of persons) the undermentioned have written to me consenting to become members of the proposed Society :

Messrs :

R. N. Apte, M.A., LL. B., F.R A.S., Prof. of Math., Rajaram College, Kolhapur.

M. V. Arunachala Sastri, M. A., Assistant Professor, Nizam's College, Hyderabad.

K. Chinnatambi Pillai, B.A., Assistant Professor, Christian College, Madras.

B. Hanumanta Rao, B.A., Prof. of Mathematics, College of Engineering, Madras.

D. K. Hardikar, B.A., Professor of Mathematics, Nizam's College, Hyderabad.

G. Kasturiranga Aiyangar, M.A., Lecturer, Maharajah's College, Mysore.

B. Krishnamachari, M.A., Asst. Superintendent, Accountant-General's Office, Madras.

A. V. Kuttikrishna Menon, M. A., Teachers' College, Saidapet.

V. Madhava Rao, M.A., Prof. of Mathematics, Maharaja's College, Vizianagaram.

M. T. Narayana Aiyangar, M A, Prof. of Mathematics, Central College, Bangalore.

R. P. Paranjpye, B. sc., M.A., Principal and Professor of Mathematics, Fergusson College, Poona.

R. Ramachandra Rao, B. A., Collector of Kurnool.

K. J. Sanjana, M.A., Principal and Professor of Mathematics, Samaldas College, Bhavnagar, Kathiawar.

P. V. Seshu Aiyar, B.A., Lecturer, Government College, Kumbhakonam.

S. P. Singaravelu Mudaliar, B.A., Asst. Professor of Mathematics, Christian College, Madras.

R. Swaminatha Aiyar, B.A., Treasury Deputy Collector, Coimbatore.

T. R. Venkataswami Nayudu, B.A., Professor of Mathematics, Maharaja's College, Mysore.

K. Krishnan Nayar, B.A., B.C.E., District Board Engineer, Mangalore.

S. A. Subramania Aiyar, B.A, B.C.E., Executive Engineer, Madras P. W. D., Madanapalli.

With me it makes 20 members.

3. I beg to declare on behalf of all those that have joined, that the Society is now formed, under the proposed name "The Analytic Club" for the time being; and I shall be its Secretary provisionally.

4. My thanks are due to the gentlemen who have joined. for the support they have given me in starting the Club. The membership has already exceeded my modest anticipations, and many more, I think, will be joining. This renders some changes and a better organization at once necessary. I shall soon be submitting to members proposals for a simple Constitution for the Society according to which the affairs of the Society will be managed by a committee consisting of a President, a few

office-bearers and some additional members. From the support that I have received in this respect also, I feel we shall have a Committee giving the greatest possible confidence.

5. I think, the Society should now aim at being a combination of all Professors, Assistant Professors and Lecturers, and men of like interest in Mathematics or in the promotion of Mathematical study, in whatever province of India, with a view to the creation of a Central Mathematical Library from which members can obtain all mathematical, including astronomical, books and journals through the post. Every journal should be sent in circulation to members who have made a general application for the same; and every book and journal should be sent to members on special application, as often as applied for. All this, of course, subject to rules that shall be made in this behalf.

6. If this conception of the immediate aims of the Society be approved, our Library should preferably be in a place which is postally a good centre for all India. In this respect, Poona is, next to Bombay, the most central place for all India. It is practically Bombay as regards the rest of India and it is a centre as regards Bombay Presidency itself. Further, I am glad to be able to mention that Prof. Paranjpye will be willing to take charge of the Library, provision being made for the discharge of purely mechanical work through Assistant Librarians. My draft constitution will be found to provide for this. I trust Madras members will endorse my suggestions and show their full appreciation of the reasons which make Poona, which contains that noble institution, the Fergusson College, so felicitous a centre for our Society. Further, by voting Poona as our centre, we Madrassees will convince the rest of India that we do not look at the matter from a purely provincial point of view. And it is my deep hope that a combination formed on these lines may be of fruitful consequences in the future.

7. Further, we shall have to get more than one copy of most books, and journals, and probably several copies of some journals, for the sake of convenience. This being the case, some of the extra copies we get can be placed in Branch Libraries in Madras, Allahabad, &c., as membership extends, so that members in these Provinces may get them quicker on their application.

8. The subscription of Rs. 25 has been felt heavy by some of those whom I addressed. Among the provisions in the draft constitution that I shall submit, will be one enabling the Committee to reduce the subscription in the case of any particular member to Rs. 15, and to continue this concession as long as they deem fit. I trust that this provision will be passed and that it will sufficiently meet all cases where such encouragement may be necessary. The scheme ought to be taken advantage of, more specially by the younger gentlemen interested in mathematics, many of whom will be the future occupants of the mathematical chairs of our colleges,—and I hope none of them will feel deterred from applying for membership at once.

9. Intending members will communicate with the undersigned.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts of the Society for the year 1918.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
Balance from 1917	2,997	4	4	Ordinary working expenses	231	5	9
Subscriptions from Members	2,457	4	0	Books and Journals	443	3	9
Subscriptions for Journal	212	0	0	Printing Journal	640	2	0
Interest on Investments	111	0	3	Library	286	4	0
Miscellaneous	33	10	0	Closing Balance	4,210	3	1
Total ...	5,811	2	7	Total ...	5,811	2	7

I have examined the Treasurer's books and vouchers, as also the monthly Statements submitted by the Secretary, Assistant Secretary and Assistant Librarian and declare the above accounts to be correct.

N. RANGANATHAN,
Auditor.

5*th April* 1919.

MADRAS,
22*nd March* 1919.

S. NARAYANA AIYAR,
Honorary Treasurer.

Conics with vanishing Θ, Θ'.

R. VYTHYNATHASWAMY.

[*Presented to the Second Indian Mathematical Conference, January*, 1919.]

§ I. *Self Harmonic Quartics.*

Two quartics $(a_0a_1a_2a_3a_4)(x,1)^4$, $(b_0b_1b_2b_3b_4)(x,1)^4$ are said to be harmonic, if

$$a_0b_4-4a_1b_3+6a_2b_2-4a_3b_1+a_4b_0=0.$$

A quartic is self-harmonic, if

$$a_0a_4-4a_1a_3+3a_2^2=0, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

i.e. if its I—invariant vanishes.

The relation (1) is of the second degree in the coefficients. *Hence when* $\alpha\ \beta\ \gamma$ *three of the roots of a self harmonic quartic are known, there are two values* $\delta\ \delta'$ *(say) of the fourth root.*

Theorem I. *The roots of the Hessian of the cubic whose roots are* α, β, γ *are* δ, δ'.

Let the co-ordinates of α, β, γ be 1, ω, ω^2, so that the roots of the Hessian are $0,\infty$. If λ be the fourth root, we find, on equating to zero the I—invariant of $(x^3-1)(x-\lambda)$, that $\lambda=0$ or ∞. This proves the theorem.

Note. δ,δ' may be termed the cyclic centres of $\alpha\beta\gamma$. This has already been done in my Note on 'Pascal Hexagon' (J. I. M. S. Vol. X, p. 342.

Theorem II. *If* δ *be a cyclic centre of* α, β, γ, *every one of the numbers* $\alpha, \beta, \gamma, \delta$ *is a cyclic centre of the remaining three.*

For the quartic $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ is self-harmonic and the self-harmonic relation is a symmetric one as among $\alpha\ \beta\ \gamma\ \delta$.

Theorem III. *If* $\alpha\beta\gamma\delta$ *is a self-harmonic quartette, (so that each of the numbers is a cyclic centre of the other three) and if the other cyclic centres of* $\beta\gamma\delta$, $\alpha\gamma\delta$, $\alpha\beta\delta$, $\alpha\beta\gamma$ *be* $\alpha'\beta'\gamma'\delta'$ *respectively, then* (1) $\alpha'\beta'\gamma'\delta$, *is a self-harmonic quartette*; (2) *the other cyclic centres of* $\beta'\gamma'\delta'$, $\alpha'\gamma'\delta''$ $\alpha'\beta'\delta'$, $\alpha'\beta'\gamma'$ *are* $\alpha\beta\gamma\delta$ *respectively* (3) $\alpha\beta\gamma\delta$ *and* $\alpha'\beta'\gamma'\delta'$ *have the same automorphic linear group.*

As before we choose the co-ordinates of α,β,γ, as 1, ω, ω^2. Let δ be 0 so that δ' is ∞. By forming the Hessians, we readily find $\alpha'\beta'\gamma'$ to be $-\frac{1}{2}, -\frac{\omega}{2}$ and $-\frac{\omega^2}{2}$. Write these values as per following scheme

α	β	γ	δ
1	ω	ω^2	0
$-\frac{1}{2}$	$-\frac{1}{2}\omega$	$-\frac{1}{2}\omega^2$	∞
α'	β'	γ'	δ'.

Since ∞ is obviously a cyclic centre of

$$-\frac{1}{2},\ -\frac{\omega}{2_1},\ -\frac{\omega^2}{2},$$

it follows that $\alpha'\beta'\gamma'\delta'$ is self-harmonic. Further since the other cyclic centre of $\alpha'\beta'\gamma'$ *i. e.* $\left(-\frac{1}{2}, -\frac{\omega}{2}, \frac{\omega^2}{2}\right)$ is δ (C), we may either verify directly, or preferably predict from symmetry, that the other cyclic centres of $\alpha'\beta'\delta'$, $\alpha'\gamma'\delta'$, and $\beta'\gamma'\delta'$ are γ, β and α respectively. Finally to prove (3) we remark that the process by which we derive $\alpha'\beta'\gamma'\delta'$ from $\alpha\beta\gamma\delta$ involves only the Hessian and the self-harmonic relation, and is consequently invariantic for linear transformations. Hence, if any linear transformation carries the group $(\alpha\beta\gamma\delta)$ into itself, it cannot but transform $(\alpha'\beta'\gamma'\delta')$ also into itself.

Cor. 1. Associated with every self-harmonic quartette there is another and the two are symmetrically related.

Cor. 2. One cross-ratio of a self-harmonic quartette is an imaginary cube root of -1. *Hence any two self-harmonic quartettes are projective.*

Cor 3. Let $(\alpha\delta)$, $(\alpha'\delta')$ be corresponding pairs selected respectively from a self-harmonic quartette and its associated quartette. If α'' be the harmonic conjugate of α' *w.r.t.* $(\delta\delta')$, the pairs $(\alpha\delta)$, $(\alpha''\delta')$ are harmonic.

This is apparent if we take the special coordinates exhibited in the scheme.

Cor. 4. From the known property of the Hessian, we see that if $(\alpha\alpha')$ be a corresponding pair of two associated self-harmonic quartettes, the homographic transformation of period three whose fixed points are $(\alpha\alpha')$ carries $(\beta\gamma\delta)$ into itself and $(\beta'\gamma'\delta')$ into itself.

Theorem IV. Given the linear group of a quartic (*i.e.* a group consisting of identity and three mutually harmonic involutions) the quartic belongs to a one-parameter family. Three of the quartics of the family are perfect squares. If the parameters of these be the roots of the cubic $F(\lambda)=0$, the roots of the Hessian of F are the two self-harmonic associated quartics of the family.

Proof. Let the involutions of the given group have fixed points $(0,\infty)$, $(1,-1)$, $(i,-i)$, so that they are $-x, 1/x, -1/x$ It is obvious that every quartic belonging to this group must be a reciprocal quadratic in x^2, *i.e.* of the form

$$x^4+2\lambda x^2+1.$$

λ may be conveniently taken as the parameter of the quartic. There are evidently three values of λ for which the quartic is a perfect square, *viz.* ∞, 1, -1. The values of λ for which the quartic is self-

harmonic are $\pm \sqrt{3}\, i$. It may be inferred that the quartics determined by these two values are associates since they belong to the same group.

It is easily verified that $\pm \sqrt{3}\, i$ are the cyclic centres of $(\infty, 1, -1)$.

§ II. *Properties of the Polar Line of a* $\triangle$ *w. r. t. a Circum-conic.*

The cyclic centres of the vertices of an inscribed $\triangle$ are the intersections of the conic with its polar line (vide: 'Pascal Hexagon,' *loc. cit.*). The theorems of the previous paragraph translate themselves into the following geometrical properties.

(1) Let the polar line of ABC be D D', (all the letters referring to points on the conic). Then the polar lines of BCD, ACD, ABD pass respectively through A, B and C. Let these polar lines be AA', BB', CC'.

(2) The polar lines of B'C'D', A'C'D', A'B'D', A'B'C' are AA' BB', CC', DD' respectively.

(3) The cross ratios (A B C D), (A'B'C'D') on the conic are $-\omega$ $-\omega^2$.

(4) There is a conic touching the tangents at A B C D and having A B C D as a self-polar quadrangle. This conic passes through A'B'C'D' and its Θ, Θ' invariants with the given conic both vanish.

These follow immediately from the geometric interpretation of harmonic quartics quoted in Grace and Young's Algebra of Invariants. These results are proved below by elementary methods.

(5) The conic inscribed in corresponding triangles BCD, B'C'D' (say) has double contact with the given conic the chord of contact being AA'. (cor 4),

(6) AA', BB', CD', C'D touch a conic having double contact with the given conic. (cor 2).

(7) Let D'L, D'M, D'N be drawn conjugate to DA, DB, DC respectively. LA', LB', LC' are all conjugate to DD' and are therefore *concurrent* at the pole of DD' (cor. 3).

(8) ABCD and A'B'C'D' have the same harmonic triangle (Th. III.)

§ III. *The Four-point System of Conics.*

It can be shewn that every four-point system contains a unique pair of conics for which Θ, Θ' vanish. If these be S_1, S_2, the values of λ for which $S_1 + \lambda S_2$ is a line conic are the roots of an equation of the form $\lambda^3 + K = 0$ since $\Theta\,\Theta'$ vanish. Now the values of λ for which $S_1 + \lambda S_2$ is S_1 or S_2 are $0, \infty$. Hence we have the important theorem.

The cyclic centres of the three line-conics of a four-point system are the conics for which Θ, Θ' *vanish.*

Now let us choose a different co-ordinate system for the conics, by taking the cross ratio of the four common points in an assigned order on each of the conics as its parameter. The line-conics are represented in this system by the parameters 0, ∞, 1. Hence the parameters of the two conics for which Θ, Θ' vanish are $-\omega, -\omega^2$. Hence,

The four points of intersection of two conics for which Θ, Θ' *vanish form a self-harmonic quartette on either ; and conversely.*

Since any two conics can be reciprocated into each other, we have the perfectly general theorem.

The cross ratio of the points of interseotion of two conics on one of them is equal to the cross of the points of contact of the common tangents on the other.

Hence the points of contact of common tangents on either of two conics for which Θ, Θ' vanish, form a self-harmonic quartette.

Now if A B C D be the points of intersection A′B′C′D′ the points of contact of common tangents on one of the conics, then evidently A B C D and A′B′C′D′ have the same harmonic triangle. Hence

The points of intersection and the points of contact of common tangents on either of two conics for which Θ, Θ' *vanish, form associated self harmonic quartettes.*

The converse of this may be proved by using the general cross-ratio theorem.

Bernoulli's Polynomials and Fourier's Series.

By S. R. Ranganathan.

[The object of this note is (i) to establish the Fourier expansions of Bernoulli's Polynomials and (ii) to deduce the sums of certain interesting types of series in a finite form and to work out some illustrative examples with the help of these formulae. The results (1·4), (2·4) and (1·6) seem to be particularly interesting.

Of these (i) is suggested as a problem in Whittaker and Watson's *Modern Analysis*; the other references, which are all to Bromwich's *Infinite Series* are given as they occur.]

In what follows, series of the type

$$\sum \frac{\sin 2\pi n x}{n^p} \text{ and } \sum \frac{\cos 2\pi n x}{n^p}$$

occur frequently. Hence, it would be well to observe at the beginning, that both the above series are absolutely convergent for all values of $p>1$; they are also easily shown, with the help of Weierstrass's M-test to be uniformly convergent for all values of x, provided $p>1$ (*vide*: Bromwich, Chap. VII, Art. 44, Ex. 1.). Further,

$$\sum \frac{\sin 2\pi n x}{n}$$

is uniformly convergent in the open interval (0, 1).

Starting with the well-known result (Bromwich, Chapter VIII, Art. 65)

$$\frac{-1}{\pi}\left(\frac{\sin 2\pi x}{1}+\frac{\sin 4\pi x}{2}+\frac{\sin 6\pi x}{3}+\ldots\right)=x-\tfrac{1}{2}, \text{ if } 0<x<1$$

and, integrating term by term, we get

$$\frac{(-1)^2}{2\pi^2}\left(\frac{\cos 2\pi x}{1^2}+\frac{\cos 4\pi x}{2^2}+\frac{\cos 6\pi x}{3^2}+\ldots\right)=\frac{x^2}{2!}-\frac{x}{2}+\frac{B_1}{2!}.$$

Hence the constant of integration is determined to be $\frac{(-1)^2 B_1}{2!}$, by putting $x=0$ on both sides and remembering that

$$\frac{1}{2^{2s-1}\pi^{2s}}\sum\frac{1}{r^{2s}}=\frac{B_s}{(2s)!}$$

where B_s is the sth. Bernoullian number, s, being an integer.

Similarly, integrating the first result $(2n-1)$ times, we get

$$\frac{(-1)^{n+1}}{2^{n-1}\pi^{2n}}\left\{\frac{\cos 2\pi x}{1^{2n}}+\frac{\cos 4\pi x}{2^{2n}}+\frac{\cos 6\pi x}{3^{2n}}+\ldots\right\}$$

$$= \frac{x^{2n}}{(2n)!} - \frac{x^{2n-1}}{2\,(2n-1)!} + \frac{B_1\, x^{2n-2}}{2!\,(2n-2)!} - \frac{B_2\, x^{2n-4}}{4!\,(2n-4)!}$$

$$+ \frac{B_3\, x^{2n-6}}{6!\,(2n-6)!} - \ldots\ldots (-)^{n+1} \frac{B_n}{(2n)!}$$

$$= \frac{1}{(2n)!}\left\{ x^{2n} - \frac{(2n)\, x^{2n-1}}{2} + \frac{(2n-1)\,(2n)\, B_1\, x^{2n-2}}{2!} - \ldots \right.$$

$$\left. + (-)^{n+1} B_n \right\}.$$

$$= \frac{1}{(2n)!}\left\{ \phi_{2n}(x) + (-)^{n+1} B_n \right\}.$$

where $\phi_m(x)$ stands for the mth. Bernoulli's Polynomial, (*cf.* Bromwich. Chapter X. Art 94).

Integrating once more, we get

$$\frac{(-1)^{n+1}}{2^{2n}\pi^{2n+1}} \left\{ \frac{\sin 2\pi x}{1^{2n+1}} + \frac{\sin 4\pi x}{2^{2n+1}} + \frac{\sin 6\pi x}{3^{2n+1}} + \ldots \right\}$$

$$= \frac{1}{(2n+1)!}\left\{ \phi_{2n+1}(x) \right\}$$

Thus, we have in the interval (0,1),

$$\frac{\cos 2\pi x}{1^{2n}} + \frac{\cos 4\pi x}{2^{2n}} + \frac{\cos 6\pi x}{3^{2n}} + \ldots$$

$$= \frac{2^{2n-1}\pi^{2n}}{(2n)!}\left\{ B_n + (-)^{n+1} \phi_{2n}(x) \right\} \quad \ldots \quad (1)$$

and

$$\frac{\sin 2\pi x}{1^{2n+1}} + \frac{\sin 4\pi x}{2^{2n+1}} + \frac{\sin 6\pi x}{3^{2n+1}} + \ldots$$

$$= (-)^{n+1} \frac{2^{2n}\pi^{2n+1}}{(2n+1)} \phi_{2n+1}(x) \quad \ldots \quad \ldots \quad (2)$$

Putting $x = \frac{1}{4}$ in (1), we get

$$\frac{2^{n-1}\pi^{2n}}{(2n)!}\left\{ B_n + (-)^{n+1} \phi_{2n}\left(\tfrac{1}{4}\right) \right\}$$

$$= -\frac{1}{2^{2n}} + \frac{1}{4^{2n}} - \frac{1}{6^{2n}} + \frac{1}{8^{2n}} - \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$= -\frac{1}{2^{2n}}\left(1 - \frac{1}{2^{2n-1}}\right)\left\{ 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots \right\}$$

$$= -\frac{2^{2n-1}-1}{2^{4n-1}} \cdot \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_n.$$

Thus, we have

$$B_n = \frac{(-)^n 2^{4n-1}}{2^{4n-1} + 2^{2n-1} - 1}\, \phi_{2n}\left(\tfrac{1}{4}\right), \quad \ldots \quad \ldots \quad (3)$$

which gives B_n in terms of the preceding B's. With the help of this formula, the first few B's are easily calculated, since the first few $\phi(\frac{1}{4})$' are easily calculated with the help of the substitution suggested in Bromwich, *viz.* $y = x(x-1)$ and $z = (x - \frac{1}{2})$.

Writing in full, we have

$$B_n=\frac{(-)^n}{2^{4n}+2^{3n}-2}\left\{(1-4n)+\frac{2n(2n-1)B_1 2^4}{2!}\right.$$
$$-\frac{2n(2n-1)(2n-2)(2n-3)B_2 2^8}{4!}+\ldots\ldots$$
$$\left.+(-)^n 2^n(2n-1)B_{n-1}2^{4n-4}.\right\}$$

Again, putting $x=\frac{1}{4}$ in (2), we get

$$\frac{(-)^{n+1}2^{2n}\pi^{2n+1}}{(2n+1)!}\phi_{2n}(\tfrac{1}{4})=1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\ldots \qquad (2\cdot1)$$
$$=\frac{\pi^{2n+1}E_n}{2^{2n+1}(2n)!},$$

where E_n is the nth. Euler's number. (*Cf.* Bromwich, Chap. X, Ex. 38.)

Thus,

$$E_n=\frac{(-)^{n+1}2^{4n+2}}{2n+1}\phi_{2n+1}(\tfrac{1}{4}), \qquad \ldots \qquad \ldots \quad (4)$$

which gives E_n in terms of the B's. Writing in full

$$E_n=(-)^n\left\{\frac{4n+1}{2n+1}-\frac{2n\,B_1}{2!}2^4+\frac{2n\,(2n-1)\,(2n-2)\,B_2\,2^8}{4!}\right.$$
$$\left.\ldots+(-)^n B_n 2^{4n}\right\}.$$

Next, putting $x=\frac{1}{3}$ in (1) and multiplying both sides by 2, we get

$$\frac{(2\pi)^{2n}}{(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}(\tfrac{1}{3})\right\}$$
$$=2\left\{\frac{1}{3^{2n}}+\frac{1}{6^{2n}}+\frac{1}{9^{2n}}+\ldots\right\}-\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\ldots\right\} \quad \ldots \quad (1\cdot2)$$
$$=\frac{1}{3^{2n-1}}\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots\right\}-\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots\right\}$$
$$=\frac{1-3^{2n-1}}{3^{2n-1}}\cdot\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n.$$

Thus,

$$B_n=\frac{(-)^n 2.3^{2n-1}}{3^{2n}-1}\phi_{2n}(\tfrac{1}{3}). \qquad \ldots \qquad \ldots \quad (5)$$

And, putting $x=\frac{1}{3}$ in (2) and multiplying both sides by $\frac{2}{\sqrt{3}}$, we get

$$\frac{1}{1^{2n+1}}-\frac{1}{2^{2n+1}}+\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}-\frac{1}{8^{2n+1}}+\frac{1}{10^{2n+1}}-\ldots$$
$$=\frac{(-)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!}\phi_{2n+1}(\tfrac{1}{3}). \quad \ldots \qquad (2\cdot2)$$

By means of this formula, the following results are easily got.—

$$\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{5^3}+\frac{1}{7^3}-\ldots\ldots=\frac{4\pi^3}{81\sqrt{3}} \qquad (2\cdot21)$$

$$\frac{1}{1^5}-\frac{1}{2^5}+\frac{1}{4^5}-\frac{1}{5^5}+\frac{1}{7^5}-\ldots\ldots=\frac{4\pi^5}{729\sqrt{3}} \qquad (2\cdot22)$$

$$\frac{1}{1^7}-\frac{1}{2^7}+\frac{1}{4^7}-\frac{1}{5^7}+\frac{7}{7^7}-\ldots\ldots=\frac{47\pi^7}{98{,}415\sqrt{3}} \qquad (2\cdot23)$$

$$\frac{1}{1^9}-\frac{1}{2^9}+\frac{1}{4^9}-\frac{1}{5^9}+\frac{1}{7^9}-\ldots\ldots=\frac{3236\pi^9}{55{,}801{,}305\sqrt{3}}. \qquad (2\cdot24)$$

Putting $x=\frac{1}{6}$ in (1), transposing and multiplying by 2, we get

$$1-\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}-\frac{1}{8^{2n}}-\frac{1}{10^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}-\ldots\ldots$$

$$=\frac{(2\pi)^{2n}}{(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}\left(\tfrac{1}{6}\right)\right\}+$$

$$\frac{2}{3^{2n}}\left\{1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\ldots\right\}$$

$$=\frac{2.\pi^{2n}}{3^{2n}(2n)!}\left\{(3^{2n}.2^{2n-1}+2^{2n-1}-1)\,B_n+\right.$$

$$\left.(-)^{n+1}3^{2n}.2^{2n-1}\phi_{2n}\left(\tfrac{1}{6}\right)\right\}\ldots \qquad (1\cdot3)$$

Also (1·2) can be written as

$$1+\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots$$

$$=\frac{(2\pi)^{2n}}{3^{2n}(2n)!}\left\{(1-3^{2n})\,B_n+(-)^n3^{2n}\phi_{2n}\left(\tfrac{1}{3}\right)\right\}. \qquad \ldots\ (1\cdot2)$$

Subtracting (1·3) from (1·2),

$$\frac{2\pi^{2n}}{3^{2n}(2n)!}\left\{1-2^{2n}.3^{2n})B_n+(-)^n2^{2n-1}3^{2n}\left[\phi_{2n}\left(\tfrac{1}{3}\right)+\phi_{2n}\left(\tfrac{1}{6}\right)\right]\right\}$$

$$=2\left\{\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{8^{2n}}+\frac{1}{10^{2n}}+\frac{1}{14^{2n}}+\frac{1}{16^{2n}}+\ldots\right\}$$

$$=\frac{2\pi^{2n}}{3^{2n}(2n)!}\left\{1-3^{2n})B_n+(-)^n3^{2n}\phi_{2n}\left(\tfrac{1}{3}\right)\right\},\ \text{by 1·2.}$$

So that, making use of (5), we get

$$\phi_{2n}\left(\tfrac{1}{6}\right)=\frac{1}{2^{2n-1}3^{2n}}\left\{3^{2n}(1-2^{n-1})\,\phi_{2n}\left(\tfrac{1}{3}\right)+(-)^n3^{2n}(2^{2n}-1)B_n\right\}$$

$$=(-)^n\frac{3^{2n-1}\,(2^{2n-1}+1)+2^{2n-1}-1}{2^{2n}\,3^{2n-1}}B_n. \qquad \ldots \qquad \ldots\ (6$$

Hence, we get from (1·3) and (6)

$$1-\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}-\frac{1}{8^{2n}}-\frac{1}{10^{2n}}+\cdots$$
$$=\frac{\pi^{2n}\,(3^{2n}-1)\,(2^{2n-1}-1)}{3^{2n}\,(2n)\,!}B_n. \qquad \ldots \quad (1\cdot4)$$

This gives for $n=1, 2, 3$ and 4 respetively,

$$1-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\ldots=\frac{2\pi^2}{27}. \qquad \ldots \quad \ldots \quad (1\cdot41)$$

$$1-\frac{1}{2^4}-\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{7^4}-\ldots=\frac{7\pi^4}{729}. \qquad \ldots \quad \ldots \quad (1\cdot42)$$

$$1-\frac{1}{2^6}-\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{7^6}-\ldots=\frac{403\pi^6}{393{,}660}. \qquad \ldots \quad \ldots \quad (1\cdot43)$$

$$1-\frac{1}{2^8}-\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{7^8}-\ldots=\frac{5207\pi^8}{49{,}601{,}160}. \qquad \ldots \quad (1.44)$$

Also, putting $x=\frac{1}{6}$ in (2) and multiplying both side by $\frac{2}{\sqrt{3}}$, we get

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}+\frac{1}{8^{2n+1}} \quad \ldots$$
$$=\frac{(-)^{n+1}(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!}\phi_{2n+1}(\tfrac{1}{6}). \qquad \ldots \quad (2.3)$$

Subtracting (2·2) from (2·3)

$$\frac{(-)^{n+1}\,(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!}\left\{\phi_{2n+1}(\tfrac{1}{6})-\phi_{2n+1}\,(\tfrac{1}{3})\right\}$$
$$=2\left\{\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}+\frac{1}{8^{2n+1}}-\frac{1}{10^{2n+1}}+\cdots\right\}$$
$$=\frac{(-)^{n+1}\,(2\pi)^{2n+1}}{\sqrt{3}\,(2n+1)!\,2^{2n}}\phi_{2n+1}\,(\tfrac{1}{3}),$$

so that $\phi_{2n+1}\,(\frac{1}{6})=\frac{2^{2n}+1}{2^{2n}}\,\phi_{2n+1}\,(\frac{1}{3})$. $\qquad \ldots \quad \ldots \quad \ldots \quad (7)$

Hence, we get from (2·3) and (7)

$$\frac{1}{1^{2n+1}}+\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}+\frac{1}{8^{2n+1}}-\cdots$$
$$=\frac{(-)^{n+1}2\pi^{2n+1}}{\sqrt{3}\,(2n+1)!}\,(2^{2n+1}+1)\,\phi_{2n+1}(\tfrac{1}{3}). \qquad \ldots \quad (2\cdot4)$$

Thus, by calculating the Bernoulli's Polynomials of argument $\frac{1}{3}$, both (2·2) and (2·4) can be evaluated. For example,

$$\frac{1}{1^3}+\frac{1}{2^3}-\frac{1}{4^3}-\frac{1}{5^3}+\ldots=\frac{5\pi^3}{81\sqrt{3}}. \qquad \ldots \quad \ldots \quad (2\,41)$$

$$\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{4^5}-\frac{1}{5^5}+\ldots=\frac{17\pi^5}{2916\sqrt{3}} \text{ (J.I.M.S.Q. 950)} \quad \ldots (2{\cdot}42)$$

$$\frac{1}{1^7}+\frac{1}{2^7}-\frac{1}{4^7}-\frac{1}{5^7}+\ldots=\frac{91\pi^7}{157{,}464\sqrt{3}} \quad \ldots \quad \ldots (2{\cdot}43)$$

$$\frac{1}{1^9}+\frac{1}{2^9}-\frac{1}{4^9}-\frac{1}{5^9}+\ldots=\frac{207{,}913\pi^9}{396{,}809{,}280\sqrt{3}}. \quad \ldots \quad \ldots (2{\cdot}44)$$

Again putting $x=\frac{1}{8}$ in (1), transposing and multiplying both sides by $\sqrt{2}$, we get

$$1-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}-\frac{1}{11^{2n}}-\frac{1}{13^{2n}}+\ldots$$

$$=\frac{(2\pi)^{2n}}{\sqrt{2}\,(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}\left(\tfrac{1}{8}\right)\right\}+\frac{2}{\sqrt{2}}\left\{\frac{1}{4^{2n}}-\frac{1}{8^{2n}}+\frac{1}{12^{2n}}-\ldots\right\}.$$

$$=\frac{(2\pi)^{2n}}{\sqrt{2}\,(2n)!}\left\{B_n+(-)^{n+1}\phi_{2n}\left(\tfrac{1}{8}\right)\right\}+\frac{1}{\sqrt{2}.\,2^{4n-1}}\left\{1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\ldots\right\}$$

$$=\frac{\pi^{2n}}{\sqrt{2}.\,(2n)!\,2^{4n-1}}\left\{(2^{6n-1}+2^{2n-1}-1)B_n+(-)^{n+1}2^{6n-1}\phi_{2n}\left(\tfrac{1}{8}\right)\right\} \quad \ldots (1{\cdot}5)$$

The following particular cases of (1·5) are easily calculated :—

$$1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}-\ldots=\frac{\pi^2}{8\sqrt{2}} \quad \ldots \quad \ldots \quad \ldots (1{\cdot}51)$$

$$1-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}-\ldots=\frac{11\pi^4}{768.\sqrt{2}}$$

(*cf.* Bromwich, Chap., X., Ex. 14) ... (1·52)

$$1-\frac{1}{3^6}-\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}-\ldots=\frac{2161\pi^6}{1{,}474{,}560\sqrt{2}}. \quad \ldots \quad \ldots (1{\cdot}53)$$

Similarly, we get from (2)

$$\frac{1}{1^{2n+1}}+\frac{1}{3^{2n+1}}-\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\frac{1}{9^{2n+1}}+\frac{1}{11^{2n+1}}-\ldots$$

$$=\frac{(-)^{n+1}(2\pi)^{2n+1}}{\sqrt{2}.\,(2n+1)!}\phi_{2n+1}\left(\tfrac{1}{8}\right)-\frac{2}{\sqrt{2}}.$$

$$\frac{1}{2^{2n+1}}\left\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\ldots\right\}$$

$$=\frac{\pi^{2n+1}}{\sqrt{2}\,(2n+1)!\,2^{4n+3}}\left\{(-)^{n+1}2^{6n+3}\phi_{2n+1}\left(\tfrac{1}{8}\right)-(2n+1)E_n\right\} \ldots (2{\cdot}5)$$

which gives, for example,

$$\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\ldots=\frac{3\pi^3}{64\sqrt{2}}. \quad \ldots \quad \ldots (2{\cdot}51)$$

$$\frac{1}{1^5}+\frac{1}{3^5}-\frac{1}{5^5}-\frac{1}{7^5}+\ldots=\frac{19\pi^5}{4096\sqrt{2}}. \quad \ldots \quad \ldots \quad (2{\cdot}52)$$

$$\frac{1}{1^7}+\frac{1}{3^7}-\frac{1}{5^7}-\frac{1}{7^7}+\ldots=\frac{307\pi^7}{455{,}360\sqrt{2}}. \quad \ldots \quad (2{\cdot}53)$$

$$\frac{1}{1^9}+\frac{1}{3^9}-\frac{1}{5^9}-\frac{1}{7^9}+\ldots=\frac{83{,}579\pi^9}{1{,}633{,}607{,}680\sqrt{2}}. \quad \ldots \quad (2{\cdot}54)$$

Lastly, adding (1·2) and (1·4) side by side, we get, after slight simplification,

$$1+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\frac{1}{17^{2n}}+\frac{1}{19^{2n}}+\ldots$$
$$=\frac{\pi^{2n}(3^{2n}-1)\,(2^{2n}-1)}{2.\,3^{2n}\,(2n)!}\,B_n. \quad \ldots \quad (1{\cdot}6)$$

Thus,

$$1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\ldots=\frac{\pi^2}{9}, \quad \ldots \quad \ldots \quad (1{\cdot}61)$$

$$1+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\frac{1}{13^4}+\ldots=\frac{5\pi^4}{486}. \quad \ldots \quad \ldots \quad (1{\cdot}62)$$

$$1+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\frac{1}{13^6}+\ldots=\frac{7\pi^6}{7290}. \quad \ldots \quad \ldots \quad (1{\cdot}63)$$

Similarly we get from (2·2) and (2·4),

$$\frac{1}{1^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}-\frac{1}{11^{2n+1}}+\frac{1}{13^{2n+1}}-\frac{1}{17^{2n+1}}+\frac{1}{19^{2n+1}}-\ldots$$
$$=\frac{(-)^{n+1}\,2^{2n}\pi^{2n+1}}{\sqrt{3}(2n+1)!}\left\{\phi_{2n+1}\left(\tfrac{1}{6}\right)+\phi_{2n+1}\left(\tfrac{1}{3}\right)\right\}$$
$$=\frac{(-)^{n+1}\pi^{2n+1}(2^{2n+1}+1)}{\sqrt{3}\,(2n+1)!}\,\phi_{2n+1}\left(\tfrac{1}{3}\right). \quad \ldots \quad \ldots \quad (2{\cdot}6)$$

Thus,

$$\frac{1}{1^3}-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}-\ldots=\frac{\pi^3}{18\sqrt{3}} \quad \ldots \quad \ldots \quad (2{\cdot}61)$$

$$\frac{1}{1^5}-\frac{1}{5^5}+\frac{1}{7^5}-\frac{1}{11^5}+\frac{1}{13^5}-\ldots=\frac{11\pi^5}{1944\sqrt{3}} \quad \ldots \quad \ldots \quad (2\,62)$$

$$\frac{1}{1^7}-\frac{1}{5^7}+\frac{1}{7^7}-\frac{1}{11^7}+\frac{1}{13^7}-\ldots=\frac{301\pi^7}{524{,}880\sqrt{3}} \quad \ldots \quad \ldots \quad (2{\cdot}63)$$

$$\frac{1}{1^9}-\frac{1}{5^9}+\frac{1}{7^9}-\frac{1}{11^9}+\frac{1}{13^9}-\ldots=\frac{15371\pi^9}{264{,}539{,}520\sqrt{3}}. \quad \ldots \quad (2{\cdot}64)$$

Note :—The calculation of $\phi_{2n+1}\left(\frac{1}{3}\right)$ and $\phi_n\left(\frac{1}{3}\right)$ which are required in the above can be done almost mentally up to ϕ_9 by making use of the substitution given in Bromwich, *viz.*, $y=x(x-1)$ and $z=x-\frac{1}{2}$. Hence it is that the formulae are interesting for application to simple numerical cases.

Legitimacy of the ordinary Complex Numbers.

By G. A. Miller.

It is well known that in 1797 Caspar Wessel presented to the Academy of Copenhagen a memoir in which he established the legitimacy of the ordinary complex numbers by giving a geometric meaning to these numbers and to their combination by addition, subtraction, multiplication and division. Somewhat later W. R. Hamilton (1837), and A. L. Cauchy (1847), established this legitimacy in two additional ways, the former having employed for this purpose the now classic method of real number couples while the latter employed the less commonly used method based on equivalence with respect to modulus i^2+1. C. F. Gauss had stated that he would publish a demonstration of the legitimacy, but he failed to do so.

In view of the great usefulness of the ordinary complex numbers it may be desirable to give here a proof of their legitimacy which differs very materially from those noted above but is based on facts which are very well known. This proof is here based largely on the group concept but it can clearly be given without the explicit use of this concept.

It is evident that the totality of the complex numbers $a+b\,i$, where a and b are real numbers and $i=\sqrt{(-1)}$ can be placed in a (1, 1) correspondence with the translations in the plane represented by the equations

$$x'=x+a$$
$$y'=y+b.$$

In fact, we may say that each of these translations is a complex number and *vice versa*. By doing this, these complex numbers become dynamic (and not static) geometric elements.

The two complex numbers

$$a+bi \text{ and } c+di$$

or the two translations

$$x'=x+a \qquad x'=x+c$$
$$y'=y+b, \qquad y'=y+d,$$

are clearly equivalent to the single translation

$$x'=x+a+c$$
$$y'=y+b+d$$

That is, when the complex numbers are regarded as these translations, two of them are equivalent to their sum and this proves that the ordinary complex numbers can be combined according to addition, reducing to the addition of reals when the numbers are real. The tota-

* Cf. *Encyclopedie des Sciences Mathematiques*, t. 1, vol. 1, p. 339.

ity of the complex numbers thus constitutes a continuous two parameter group, viz., the group of translations in the plane. This group is abelian and its identity is the ordinary zero.

The complex number $a+bi$ may also be said to be the linear transformation represented by the equations*

$$x' = ax - by$$
$$y' = bx + ay$$

If these complex numbers with the exception of 0 are represented by the totality of such linear transformations, it is easy to see that two complex numbers are equivalent to their product. In fact, if

$$z = x + iy \text{ and } z' = x' + iy'$$

then $z' = (a+bi)z$. Hence the ordinary complex numbers, with the exception of the identity of addition, constitute another two parameter continuous group of transformations. This group exhibits the fact that these numbers can be combined by multiplication and gives a perfectly clear geometric significance to such combinations.

Although 0, or the identity of the translations group, does not belong to the latter group of transformations, we may clearly require that when 0 or the transformation

$$x' = 0x - 0y$$
$$y' = 0x + 0y$$

is combined with any of the transformations in this group it shall reduce to this tranformation. If this is done, it results that the ordinary complex numbers become real and their legitimacy has been fully established, since the said transformations are evidently legitimate.

An advantage attained by regarding the ordinary complex numbers as such transformations is that it furnishes a much broader notion of number and it puts life into numbers. The static geometric interpretation of these numbers brought them from the ghost world into the world of reals, while the dynamic geometric interpretation endows them with life and the power to do things. If one complex number is regarded as belonging to one of the two continuous groups of transformations mentioned above, then all the other complex numbers should be regarded temporarily as belonging to the same group and to be affected by the first number. Our emphasis is, however, not on this evident interpretation but on the fact that it establishes the legitimacy of the ordinary complex numbers in an interesting and useful manner.

In the *Encyclopedie des Sciences Mathematiques*, tome 1, volume I, page 362, certain groups of transformations are represented by ordinary complex numbers. The two groups noted above would seem to have

* Cf. Pascal's *Repertorium des hoheren Mathematik*, vol. 2 (1910), p. 167.

close contact with those considered at this place in the *Encyclopedie*. In fact, these groups might be regarded as among the simplest and most interesting illustrative examples of continuous transformation groups, so that the present proof of the legitimacy of the ordinary complex numbers may also serve as an easy introduction to some elements of the group concept.

UNIVERSITY OF ILLINOIS.
October, 1918.

Astronomical Notes.

1. The Orbit of Sirius.

The micrometer measures of the companion now extend over more than a complete circuit. The independent investigations of the Orbit have been made by Dr. Aitken and M. Jonkheere respectively by discussing all the available measures and there is a remarkable agreement between the two results. The elements obtained by Dr. Aitken are as follows :—

Period		50.04 years
Time of Periastron passage	...	1894.133
Eccentricity	...	0.5945
Semimajor axis	...	7″.570
Inclinatiod	...	43°.31
Nodal Point	...	42°.71
Angle between Note & Periastron.		145°.69

2. A Total Eclipse of the Sun.

A total eclipse of the Sun (invisible in India) will occur on May 29. The path of totality passes through Central Africa and crossing the Atlantic, passes through S. America. The duration of totality is 6^{m} 50^{s} in mid Atlantic. The eclipse occurs in a region of the sky which is exceptionally rich in bright stars and the Astronomer Royal has drawn attention to the importance of the occasion as it affords a favourable opportunity for testing Einstein's theory of relativity, according to which a ray of light passing near the Sun will be deflected, the deflection being inversely proportional to the distance from the Sun's centre. It is proposed to take photographs at the time of the eclipse showing a good number of stars and if successful photographs are taken, these will be compared with those of the same region already taken for the purpose, which show the same stars in their undisturbed positions. The comparison is expected to throw some light on the validity of Einstein's theory.

3. Photo-Electric observations of the brightness of the Corona were made by Prof. Stebbins during the total eclipse of June 8, 1918. The light was found to be equal to 0.6 of the light of the full Moon.

4. Comet 1786 II :—Miss. Palmer has, in Astronomical Journal No. 744, investigated the orbit of the comet 1786 II which was dis-

covered by Miss. Caroline Herschel and was observed for a period of 82 days by several astronomers. She deduces the most probable period to be 9373 years!

5. *Parallax of Nova Persei*:—From the motion of the nebulosity ronnd this Nova, Prof. Turner has determined its parallax and the value obtained is $+0''.0093$. A new determination by the direct method has been made by A. Van Maanen with the 60-inch reflector of the Mount Wilson Solar Observatory and the relative parallax deduced from 14 exposures is $+0''.007 \pm 0''.004$ which corresponds to an absolute parallax of $+0''.009$. The close agreement of the values obtained by the two different methods is noteworthy.

6. *A New Publication.* Since the late Prof. Watson's classic treatise on "*Theoretical Astronomy*" went out of print, there has been a great need for a text-book in English dealing with determination of orbits and celestial mechanics generally. The publication of Prof. Plummer's "*Introductory Treatise on Dynamical Astronomy*" admirably satisfies this need. The book will serve as a good introduction to the treatise of Tisserand and other standard works on the subject.

7. In the "Observatory" February 1919, attention is drawn to a slip in Sir R. Ball's Spherical Astronomy. The last equation on page 338, giving $\tan \frac{h}{2}$ (Semidiurnal arc) should read

$$\tan \frac{h}{2}$$

$$= \pm \{ \sin \tfrac{1}{2}(z+\phi-\delta) \sin (z-\phi+\delta) \sec \tfrac{1}{2}(z+\phi+\delta) \sec \tfrac{1}{2}(z-\phi-\delta) \}^{\frac{1}{2}}.$$

T. P. Bhaskara Sastri.

Supplementary Notes.

(1) *Statistical Mechanics in Astronomy.* It is well known that statistical mechanics owes its origin to the investigation of the laws of thermodynamics on mechanical principles. Clausius and Maxwell may be said to be the founders of the science. The method was greatly developed by Maxwell and Boltzmann.

On account of the elegance and simplicity of its principles, the science of statistical mechanics received an independent development apart from the field of thermodynamics from contributors since the time of Clausius and Maxwell, but it was chiefly known as the theory of gases.

It was proposed by Poincare to apply the theory of gases to a stellar system. But a difficulty was felt. In gas-dynamics the prominent feature is the collisions of molecules, whereas in the stellar system, collisions are exceedingly rare and cannot be the harmless rebound of the theory of gases.

During the last few years, the general methods of statistical mechanics have been brought into use for astronomical purposes and we have thus a regular progression through rigid dynamics, hydrodynamics, and gas-dynamics to stellar dynamics. Some of the prominent workers in this field of inquiry are Schwarzschild, Eddington, Jeans and Charlier. Charlier's theory lays importance on the small deflections of stars by chance encounters, and he finds that the stellar universe has approached considerably towards statistical equilibrium.

(2) *Question* 841. On p. 79 (J. I. M. S., 1917) there is a note on Question 841, stating an elegant, though rather artificial, Oxford (1898) rider, which runs thus :—

"If the moon were to move in a circular orbit round the Earth, in a plane inclined at an angle ω to the ecliptic, the time of revolution being equal to one sidereal day, show that it appears to an observer on the earth to oscillate in a slender figure of eight, over a point on the equator."

The solution may be given thus :—

Let l' be the orbital co-ordinate and R', δ' the equatorial co-ordinates of the moon referred to the node of the orbit on the equator and let ω' be the inclination of the orbit to the equator.

Since l' is the orbital co-ordinate of the moon, it is also the equatorial co-ordinate of a suitably chosen point on the earth's equator, as the two angular velocities are the same. The R' and δ' of the moon are given by the usual formulae :

$$\tan R' = \tan l' \cos \omega',$$
$$\sin \delta' = \sin l' \sin \omega'. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Hence the hour angle measured westward from the meridian through the fixed point in question is

$$h = l' - R'.$$

From this

$$\tan h = \frac{\tan l' . 2 \sin^2 \frac{\omega'}{2}}{1 + \tan^2 l' \cos \omega'}. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Equations (1) and (2) give the apparent motion as l' varies through 360°. From (1) δ' varies from 0 to ω', ω' to 0, 0 to $-\omega'$, $-\omega'$ to 0 as l' goes through the four quadrants; and from (2) $\tan h$ vanishes at $0, 90^\circ, 180^\circ, 270^\circ$ and is of alternate signs in the four quadrants and has a turning value $\pm \sin^2 \frac{\omega'}{2} \sqrt{\sec \omega'}$ (in each quadrant), which is small, so that the figure is that of 8.

R. N. Apte.

SOLUTIONS.

Question 962

(A. C. L. WILKINSON) : —If two circles are such that hexagons can be inscribed in the one which are circumscribed to the other, prove that the sum of the products of the diagonals taken two at a time is constant and equal to

$$\frac{4R^2r^2\{(R^2-c^2)^2+4R^2r^2\}}{(R^2-c^2)^3}.$$

Solution by F. H. B. Gulasekharam, M.A.

Let two fixed circles of radii R and r respectively with their centres distant c apart be such that a hexagon $A_1A_2A_3A_4A_5A_6$ is inscribed in the former and circumscribed to the latter.

(*a*) *Then the triangles* $A_1A_3A_5$ *and* $A_2A_4A_6$ *are circumscribed to the same fixed circle, coaxial with the two given circles.*

For, the variable triangle $A_1A_2A_3$ is inscribed in a fixed circle, while two of its sides touch another fixed circle ; hence, by Poncelet's Theorem, the side A_1A_3 touches a fixed circle coaxial with the two given circles. Hence, the sides A_1A_3, A_3A_5, A_5A_1, A_2A_4, A_4A_6, A_6A_2 all touch a fixed circle coaxial with the two given circles.

(*b*) *The diagonals* A_1A_4, A_2A_5, A_3A_6 *intersect in* L, *the limiting point of the system of circles coaxal with the two given circles.*

For, applying Poncelet's Theorem to the variable quadrilateral $A_1A_2A_3A_4$, the side A_1A_4 touches a fixed circle coaxial with the two given circles. Hence A_1A_4, A_2A_5 and A_3A_6 touch the same fixed circle. But by Brianchon's theorem, the three diagonals A_1A_4, A_2A_5, A_3A_6 meet in a point. Hence their point of intersection L is a limiting point of the system of circles coaxal with the two given circles.

(*c*) If t_r be the length of the tangent to the inner given circle from the vertex A_r ($r=1, 2, 3, 4, 5, 6$), then

(i) $t_1t_4=t_2t_5=t_3t_6=$ a constant,

(ii) $t_1t_3+t_3t_5+t_5t_1=t_2t_4+t_4t_6+t_6t_2=$ a constant,

(iii) $t_1t_2+t_2t_3+t_3t_4+t_4t_5+t_5t_6+t_6t_1=$ a constant.

(i) For, from the property of coaxal circles, (Theor. (4). p. 350, Nixon's *Euc. Revised*),

$$\frac{t_1}{A_1L}=\frac{t_2}{A_2L}=\frac{t_3}{A_3L}=\frac{t_4}{A_4L}=\frac{t_5}{A_5L}=\frac{t_6}{A_6L}=\text{a constant}, \quad \ldots \quad (W)$$

also $A_1L.A_4L = A_2L.A_5L = A_3L.A_6L =$ a constant.

Hence $t_1t_4 = t_2t_4 = t_5t_6 =$ a constant.

(ii) If q_r be the length of the tangent to the incircle of $\triangle A_1A_3A_5$, or of $\triangle A_2A_4A_6$), from the vertex A_r, we have

$$\frac{q_1q_3q_5}{q_1+q_3+q_5} = \rho^2,$$

where ρ is the incircle of $\triangle A_1A_3A_5$ or of $\triangle A_2A_4A_6$; and

$$\frac{(q_1+q_3)(q_3+q_5)(q_5+q_1)}{q_1+q_3+q_5} = 4R\rho.$$

Hence $$\rho^2 + 4R\rho = q_1q_3 + q_3q_5 + q_5q_1.$$

We have shewn that ρ is constant.

Hence $q_1q_3 + q_3q_5 + q_5q_1$ is constant.

Again from the property of coaxal circles,

$$\frac{q_1}{t_1} = \frac{q_2}{t_2} = \frac{q_3}{t_3} = \frac{q_4}{t_4} = \frac{q_5}{t_5} = \frac{q_6}{t_6} = \text{ a constant.}$$

$\therefore$ $$t_1t_3 + t_3t_5 + t_5t_1 = \text{ a constant.}$$

Similarly $$t_2t_4 + t_4t_6 + t_6t_2 \text{ is constant.}$$

(iii) We have $\dfrac{q_1q_3q_5}{q_1+q_3+q_5} = \rho^2 = \dfrac{q_2q_4q_6}{q_2+q_4+q_6}$.

$\therefore$ $$\frac{q_1q_2q_3q_4q_5q_6}{(q_1+q_3+q_5)(q_2+q_4+q_6)} = \rho^4 \text{ (a constant).}$$

Hence $$\frac{t_1t_2t_3\ t_4t_5t_6}{(t_1+t_3+t_5)(t_2+t_4+t_6)} \text{ is constant.}$$

Hence from c (i),

$$(t_1+t_3+t_5)t_2 + (t_4+t_6) \text{ is constant.}$$

Hence $$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_5 + t_5t_6 + t_6t_1 \text{ is constant.}$$

(d) $A_1A_4 \cdot A_2A_5 + A_2A_5 \cdot A_3A_6 + A_3A_6 \cdot A_1A_4$ is constant.

For, from the equations (W) above,

$$A_1A_4 \propto (t_1+t_4);\ A_2A_5 \propto (t_2+t_5)\ A_3A_6 \propto (t_3+t_6).$$

Hence $$A_1A_4 \cdot A_2A_5 + A_2A_5 \cdot A_3A_6 + A_3A_6 \cdot A_1A_4$$
$$\propto \{ (t_1+t_4)(t_2+t_5) + (t_2+t_5)(t_3+t_6) + (t_3+t_6)(t_1+t_4) \}$$

which is constant, from c (ii) and c (iii) above.

(e) The constant values in the foregoing are easily determined by taking the case in which the hexagon is symmetrical with respect to the line of centres of the two given circles. But before determining the

constants, let us establish the invariant relation connecting r, R and c for poristic hexagons.

Let $a_1a_2a_3a_4a_5a_6$ be the hexagon in the symmetrical position, the axis of symmetry being a_1a_4, the diameter of the given outer circle.

Write $\widehat{a_1a_4a_3}=\phi$; $\widehat{a_1a_4a_2}=\theta$. Then since the sides a_1a_2, a_4a_3 a_2a_3 touch the inner circle, it is easy to see that

$$r=(R+c)\cos\theta=(R-c)\sin\phi=R\cos(\theta-\phi)+c\cos(\theta+\phi).$$

Hence. $\cos\theta\cos\phi(R+c)+\sin\theta\sin\phi(R-c)=r.$

$$\therefore \quad \cos\phi+\sin\theta=1. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Now writing $\dfrac{r}{R+c}=\cos\theta=p,$

$$\frac{r}{R-c}=\sin\phi=q,$$

the relation (i) is equivalent to

$$3-2(p^2+q^2)=(p^2-q^2)^2,$$

or $$\{1+(p-q)^2\}\{1+(p+q)^2\}=4, \quad \ldots \quad \ldots \quad (2)$$

giving the invariant relation between r, R, c, for poristic hexagons.

(*f*) *To determine the consant value of*

$$A_1A_4\cdot A_2A_5+A_2A_5\cdot A_3A_6+A_3A_6\cdot A_1A_4$$
$$=a_1a_4\cdot a_2a_5+a_2a_5\cdot a_3a_6+a_3a_6\cdot a_1a_4.$$

We have $a_1a_4=2R$

$$a_2a_5=a_3a_6=2R\sin\theta+\phi.$$

Again, since from (*e*) (i)

$\cos\phi+\sin\theta=1$, and $\sin\phi+\cos\theta=p+q$,

we have $$2+2\sin(\theta+\phi)=\{1+(p+q)^2\}.$$

$$\therefore \quad a_1a_4+a_2a_5=R[1+(p+q)^2].$$

Hence, the required constant value

$$=2a_1a_4\cdot a_2a_5+(a_2a_5)^2$$
$$=(a_1a_4+a_2a_5)^2-(a_1a_4)^2$$
$$=R^2[1+(p+q)^2]^2-4R^2$$
$$=R^2[1+(p+q)^2]\left[1+(p+q^2)-\frac{4}{1+(p+q)^2}\right]$$
$$=R^2[1+(p+q)^2][1+(p+q)^2-1-(p-q)^2], \text{ from } e(2)$$
$$=4R^2pq[1+(p+q)^2]$$
$$=\frac{4R^2r^2[(R^2-c^2)^2+4R^2r^2]}{(R^2-c^2)^3}.$$

Question 963.

(A. C. L. WILKINSON) :—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, prove that the ratio of these quadrilaterals is constant.

Solution by F. H. V. Gulasekharam, M.A.

To avoid ambiguity, the latter part of the question should read "prove that the area of any such quadrilateral bears a constant ratio to the area of the corresponding quadrilateral whose angular points are the four points of contact of the inner circle and the original quadrilateral."

Let there be two circles such that a quadrilateral can be inscribed in the one and circumscribed to the other. Let $A_1A_2A_3A_4$ be one such quadrilateral; and T_1, T_2, T_3, T_4 be the points of contact of the sides A_1A_2, A_2A_3, A_3A_4, A_4A_1 respectively with the inner circle.

If ϕ be the angle between the diagonals A_1A_3 and A_2A_4, the area of the quadrilateral $A_1A_2A_3A_4 = \frac{1}{2} A_1A_3 . A_2A_4 \sin\phi$.

Again since T_1T_3 and T_2T_4 intersect at right angles, the area of the quadrilateral $T_1T_2T_3T_4 = \frac{1}{2} T_1T_3 . T_2T_4$.

(*vide* : Nixon, *Euclid Revised*, p. 187. Ex. 87)

Hence we are required to prove that

$$\frac{A_1A_3 . A_2A_4 . \sin\phi}{T_1T_3 . T_2T_4} \text{ is constant.}$$

Again we see that T_1T_3 and T_2T_4 bisect the angles between A_1A_3 and A_2A_4; and the point of intersection L of A_1A_3 and A_2A_4 is the limiting point of the system of circles coaxal with the two given circles. (Nixon : *Euclid Revised*, pp. 370, 375 and 424.)

Hence if t_r denotes the length of the tangent from the vertex A_r $(r=1, 2, 3, 4)$ to the inner circle, we have

$$\frac{t_1}{A_1L} = \frac{t_2}{A_2L} = \frac{t_3}{A_3L} = \frac{t_4}{A_4L} = \text{a constant (say) } 1/\lambda;$$

also $A_1L . A_3L = A_2L . A_4L = \text{a constant.}$

$\therefore$ $t_1t_3 = t_2t_4 = \text{a constant (say) } \mu$

Also $A_1A_3 = \lambda t(_1 + t_3)$; $A_2A_4 = \lambda(t_2 + t_4)$.

By Ptolemy's theorem,

$$A_1A_3 . A_2A_4 = A_1A_2 . A_3A_4 + A_1A_4 . A_2A_3.$$

$$\therefore \quad \lambda^2 (t_1+t_3)(t_2+t_4) = (t_1+t_2)(t_3+t_4) + (t_1+t_4)(t_2+t_3)$$
$$= (t_1+t_3)(t_2+t_4) + 4\mu.$$

Hence $(t_1+t_3)(t_2+t_4)$ is constant.

Hence $A_1A_3 . A_2A_4$ is constant. (1)

Again since LT_1 bisects the angle A_1LA_2,

$$LT_1 = 2\frac{A_1L . A_2L}{A_1L + A_2L}\cos\frac{\phi}{2}$$ (Hobson: *Plane Trig.* p. 196.)

Similarly $$LT_3 = 2.\frac{A_3L . A_4L}{A_3L + A_4L}\cos\frac{\phi}{2}$$

Hence T_1T_3 varies as $\left\{\frac{t_1t_2}{t_1+t_2} + \frac{t_3t_4}{t_3+t_4}\right\}\cos\frac{\phi}{2}$.

But $$t_1t_3 = t_2t_4 = \text{a constant.}$$

$\therefore$ $$T_1T_3 \propto \frac{(t_1+t_2+t_3+t_4)}{(t_1+t_2)\ (t_3+t_4)}\cos\frac{\phi}{2}.$$

Similarly $$T_2T_4 \propto \frac{(t_1+t_2+t_3+t_4)}{(t_1+t_4)(t_2+t_3)}\cos\frac{\pi-\phi}{2}.$$

$\therefore$ $$T_1T\ T_2T_4 \propto \sin\phi \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\left(\text{For } \frac{\{t_1+t_2+t_3+t_4\}^2}{(t_1+t_2)\ (t_2+t_3)\ (t_3+t_4)\ (t_4+t_1)} = \frac{1}{r^2}.\right)$$

(Hobson's *Plane Trig.*, p. 208)

Hence from (1) and (2)

$$\frac{A_1A_3 . A_2A_4 . \sin\phi}{T_1T_3 . T_2T_4} \text{ is constant,}$$

whence the theorem.

Question 964.

(H Br.) :—If $f(x, y)$ is a rational algebraic function of x and y such that $f(-x,y) = (-)^{k+1} f(x,y)$, show that

(i) $$\int_0^\infty f(\sin 2\theta, \cos 2\theta)\frac{d\theta}{\theta^{k+1}}$$

$$= \frac{(-)^k}{k!}\int_0^\infty f(\sin 2\theta, \cos 2\theta)\tan\theta\frac{d^k}{d\theta^k}(\cot\theta)\frac{d\theta}{\theta}$$

$$= \frac{(-)^k}{k!}\int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta)\frac{d^k}{d\theta^k}(\cot\theta)\,d\theta;$$

and (ii) $$\int_0^\infty f(\sin\theta, \cos\theta)\frac{d\theta}{\theta^{k+1}}$$

$$= \frac{(-)^k}{k!}\int_0^{\frac{1}{2}\pi} f(\sin\theta, \cos\theta)\frac{d^k\lambda}{d\theta^k}\,d\theta;$$

where $\lambda = \cot\theta$, or $\operatorname{cosec}\theta$, according as

$f(-x,-y) = f(x,y)$, or $-f(x, y.)$

[The integrals are assumed to be finite].

Solution by K. B. Madhava.

These results may be supposed to be more general than the general case considered by me in solution to Q. 702 (published on p. 190 of Vol. VIII, J.I.M. S.,) but the same method of procedure is also available here.

Since the integrals may be assumed to be finite, the range of integration may be split up into parts so that

$$\int_0^\infty = \int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi} + \int_{\pi}^{\frac{3}{2}\pi} + \int_{\frac{3}{2}\pi}^{2\pi} \quad \dots + \quad \dots \quad \dots \quad (1)$$

and the following substitutions be made:

in the second interval, put $\phi = \pi - \theta$

third ,, put $\phi = \theta - \pi$

fourth ,, put $\phi = 2\pi - \theta$, and so on (2)

We shall therefore have in the first case,

$$\int_0^\infty f(\sin 2\theta, \cos 2\theta)\frac{d\theta}{\theta^{k+1}}$$

$$= \int_0^{\frac{1}{2}\pi} d\phi \left[\frac{f(\sin 2\phi, \cos 2\phi)}{\phi^{k+1}} + \frac{(-)^{k+1} f(\sin 2\phi, \cos 2\phi)}{(\pi-\phi)^{k+1}}\right.$$

$$\left. + \frac{f(\sin 2\phi, \cos 2\phi)}{(\pi+\phi)^{k+1}} + \dots\right]$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\phi, \cos 2\phi)\ d\phi \left[\frac{1}{\phi^{k+1}} + \left\{\frac{1}{(\phi-\pi)^{k+1}}\right.\right.$$

$$\left.\left. + \frac{1}{(\phi+\pi)^{k+1}}\right\} + \dots\right]$$

$$= \frac{(-)^k}{k!} \int_0^{\frac{1}{2}\pi} f(\sin 2\phi, \cos \phi)\ \frac{d^k}{d\phi^k}(\cot \phi)\, d\phi,$$

which proves the first part.

Again, for the second part, proceeding in the same manner we notice that f repeats itself, multiplied by unity in every odd interval and by $(-)^{k+1}$ in every event one; and that $\frac{d^k}{d\theta^k}$ (cot θ) repeats itself similarly, so that in every interval we shall have $f(\sin 2\theta, \cos 2\theta)\frac{d^k}{d\theta^k}(\cot \theta)$ repeats unaltered by the substitutions (2), tan θ however changes into $(-)^n \tan \theta$.

Therefore

$$\int_0^\infty f(\sin 2\theta, \cos 2\theta) \tan\theta \frac{d^k}{d\theta^k}(\cot\theta) \frac{\theta}{d\theta}.$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta).\, d\theta \left[\frac{\tan\theta}{\theta} - \frac{\tan\theta}{\pi-\theta} + \frac{\tan\theta}{\pi+\theta} - \frac{\tan\theta}{2\pi-\theta} + \cdots\right] \frac{d^k}{d\theta^k}(\cot\theta)$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta).\, d\theta \left[\frac{\tan\theta}{\theta} + \frac{\tan\theta}{\theta-\pi} + \frac{\tan\theta}{\theta+\pi} + \frac{\tan\theta}{\theta+2\pi} \cdots + \right] \frac{d^k}{d\theta^k}(\cot\theta)$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta).\, d\theta.\, \tan\theta.\, \cot\theta.\, \frac{d^k}{d\theta^k}(\cot\theta).$$

$$= \int_0^{\frac{1}{2}\pi} f(\sin 2\theta, \cos 2\theta) \frac{d^k}{d\theta^k}(\cot\theta);$$

which establishes the first case completely.

For case (ii), we still proceed in the same manner; thus

$$\int_0^\infty f \sin\theta, \cos\theta) \frac{d\theta}{\theta^{k+1}}$$

$$= \int_0^{\frac{1}{2}\pi} d\theta \left[\frac{f(\sin\theta, \cos\theta)}{\theta^{k+1}} + \frac{f(\sin\theta, -\cos\theta)}{(\pi-\theta)^{k+1}} + \frac{f(-\sin\theta, -\cos\theta)}{(\pi+\theta)^{k+1}} + \frac{f(-\sin\theta, \cos\theta)}{(2\pi-\theta)^{k+1}} + \frac{f(\sin\theta, \cos\theta)}{(2\pi+\theta)^{k+1}} + \cdots\right]$$

$$= \int_0^{\frac{1}{2}\pi} d\theta.\, f(\sin\theta, \cos\theta) \left[\frac{1}{\theta^{k+1}} - \left(\frac{1}{(\theta+\pi)^{k+1}} + \frac{1}{(\theta+\pi)^{k+1}}\right) + \left(\frac{1}{(\theta-2\pi)^{k+1}} + \frac{1}{(\theta+2\pi)^{k+1}} - + \cdots\right)\right]$$

if $f(-x,-y) = -f(x, y)$;
the integral

$$= \int_0^{\frac{1}{2}\pi} d\theta.\, f(\sin\theta, \cos\theta) \left[\frac{1}{\theta^{k+1}} + \left(\frac{1}{(\theta-\pi)^{k+1}} - \frac{1}{(\theta+\pi)^{k+1}}\right) + \frac{1}{(\theta-2\pi)^{k+1}} - \frac{1}{(\theta+2\pi)^{k+1}}\right) + \cdots\right]$$

if $f(-x,-y) = f(x, y$;

so that the second part $= \int_0^{\frac{1}{2}\pi} d\theta .\ f\ (\sin\theta, \cos\theta)\ \frac{d^k\lambda}{d\theta^k}$

where $\lambda = \cot\theta$, or $\operatorname{cosec}\theta$ according as $f(-x,-y) = f(x, y)$ or $-f(x, y)$, as in the problem.

Question 965.

(Hemraj) :—ABC is a triangle. A point J is taken on AI produced, such that IJ subtends at B or C an angle α, where $\cot\alpha = 3\cot\frac{1}{2}C$ or $3\cot\frac{1}{2}B$. Show that there exists a circle through A and J which cuts AB and AC in D and E, so that $BD = CE = BC$. [Suggested by Q. 959]

Solution by K. J. Sanjana and N. G. Leather.

Take the figure and notation of the solution to Q. 959.

Then $\alpha = \angle JBI_1$: let $\cot a = \frac{l+m+1}{l+m-1}\cot\frac{C}{2}$, more generally. It is easily seen that $\angle BI_1I = \frac{1}{2}C$, so that $\angle BJI_1 = \frac{1}{2}C - \alpha$.

Hence $$\frac{JI_1}{I_1B} = \frac{\sin\alpha}{\sin(\frac{1}{2}C-\alpha)} = \frac{1}{\sin\frac{1}{2}C\cot\alpha - \cos\frac{1}{2}C} = \frac{l+m-1}{2\cos\frac{1}{2}C};$$

$$\therefore \quad JI_1 = \frac{l+m-1}{2\cos\frac{1}{2}C}.\ 4R\cos\frac{C}{2}\sin\frac{A}{2} = 2R\,(l+m-1)\sin\frac{A}{2}.$$

Adding I_1A, $JA = R\{\,2(l+m-1)\sin\frac{1}{2}A + 4\cos\frac{1}{2}B\cos\frac{1}{2}C\,\}$

$$= \frac{R}{\cos\frac{1}{2}A}\left\{(l+m-1)\sin A + \sin A + \sin B + \sin C\right\}$$

$$= \{\,(l+m)\,a + b + c\,\}\,/2\cos\tfrac{1}{2}A.$$

Let any circle through A and J cut AB in D and AC in E. From

$$JA.\ DE = JD.\ AE + JE.\ AD,$$

we get $$\frac{(l+m)a+b+c}{2\cos\frac{1}{2}A}.\ 2R'\sin A = 2R'\sin\frac{A}{2}(b'+c'),$$

$$\therefore \quad b'+c' = (l+m)\,a + b + c, \text{ or } (b'-b)+(c'-c) = (l+m)\,a.$$

Thus $BD + CE = (l+m).\ BC$. If $l = m = 1$, $\cot\alpha = 3\cot\frac{1}{2}C$, and $BD + CE = 2a$, so that for some position of the circle $BD = CE = a$.

The same result follows from $\cot\alpha = 3\cot\frac{1}{2}B$.

Question 965 (*a*).

(A. C. L. WILKINSON) :—Prove that

$$\frac{1-2\,(2+k^2)}{2s_1c_1d_1}\frac{s_1^2+5k^2s_1^4}{}+\frac{1-2\,(1+k^2)\;s_2^2+3k^2s_2^4}{2s_2c_2d_2}$$

$$-\frac{1-2\,(1+2k^2)\;s_3^2+5k^2s_3^4}{2s_3c_3d_3}=-3k^2s_1s_2s_3\,;$$

where $s_1=\mathrm{sn}\left(\frac{k}{3}\right)$, $s_2=\mathrm{sn}\left(\frac{ik'}{3}\right)$, $s_3=\mathrm{sn}\left(\frac{k+ik'}{3}\right)$.

Solution by F. H. V. Gulasekharam, M. A.

From the addition formula for the E functions, we have

$$3\,\mathrm{E}\left(\frac{k+ik'}{3}\right)-3\,\mathrm{E}\left(\frac{k}{3}\right)-3\,\mathrm{E}\left(\frac{ik'}{3}\right)$$

$$=-3k^2s_1s_2s_3. \qquad \ldots \qquad \ldots \quad \text{(A)}$$

We shall now establish the following relations :—

(i) $3\,\mathrm{E}\left(\frac{k+ik'}{3}\right)=\mathrm{E}+i\,(k'-\mathrm{E}')-\frac{1-2\,(1+2k^2)\;s_3^2+5k^2s_3^4}{2s_3c_3d_3}$

(ii) $3\,\mathrm{E}\left(\frac{k}{2}\right)=\mathrm{E}-\frac{1-2\,(2+k^2)\;s_1^2+5k^2s_1^4}{2s_1c_1d_1}$

(iii) $3\,\mathrm{E}\left(\frac{ik'}{3}\right)=i\,(k'-\mathrm{E}')-\frac{1-2\,(1+k^2)\;s_2^2+1k^2s_2^4}{2s_2c_2d_2}$

To prove (i).

It is easy to reduce the relation

$$3\,\mathrm{E}u=\mathrm{E}(3u)+k^2\mathrm{snu}\;\mathrm{sn}\;2u\;(\mathrm{snu}+\mathrm{sn}\;3u)$$

Now write $3u=k+ik'$, Then $\mathrm{E}(3u)=\mathrm{E}+i(k'-\mathrm{E}')$, $\mathrm{sn}\,3u=\frac{1}{k}$,

and, $\mathrm{sn}\,2u=\mathrm{sn}\left[\frac{2}{3}(k+ik')\right]=\mathrm{sn}\left[(k+ik')-\frac{1}{3}(k+ik)'\right]=\frac{1}{k}\frac{d_3}{c_3}$.

Hence $$3\,\mathrm{E}\left(\frac{k+ik'}{3}\right)=\mathrm{E}+i(k'-\mathrm{E}')+\frac{s_3d_3}{c_3}(1+ks_3)$$

$$=(\mathrm{E}+i(k'-\mathrm{E}')+\frac{2s_3^2d_3^2}{2s_3c_3d_3}\,(1+ks_3).$$

Again, since $\mathrm{sn}\left[\frac{2}{3}(k+ik')\right]=\frac{1}{k}\frac{d_3}{c_3}$,

$$\frac{2s_3c_3d_3}{1-k^2s_3^4}=\frac{1}{k}\frac{d_3}{c_3},$$

i.e. $$1-2ks_3+2\,k\,s_3^2-k^2s_3^4=0.$$

Making use of the foregoing relation

$$2s_3^2d_3^2(1+ks_3)$$
$$= 2s_3^2+2ks_3^3-2k^2s_3^4-2k^2s_3^5$$
$$= 2s_3^2+(k^2s_3^4+2ks_3-1)-2k^2s_3^4$$
$$-2s(1-2ks_3+2ks_3^3)$$
$$= -[1-2(1+2k^2)s_3^2+5k^2s_3^4].$$

Hence relation (i) is established.

To prove relation (ii).

As before, starting with the identity

$$3\,\mathrm{E}u=\mathrm{E}\,3u+k^2\,\mathrm{sn}u\,\mathrm{sn}\,2u\,(\mathrm{sn}u+\mathrm{sn}\,3u),$$

write $3u=k$; so that $\mathrm{E}\,3u=\mathrm{E}$, $\mathrm{sn}\,3u=1$,

$$\text{and, } \mathrm{sn}\,2u=\mathrm{sn}\frac{2k}{3}=\mathrm{sn}\left(k-\frac{k}{3}\right)=\frac{c_1}{d_1}$$

$$\text{Hence } 3\,\mathrm{E}\left(\frac{k}{3}\right)=\mathrm{E}+\frac{k^2s_1c_1\,(1+s_1)}{d_1}$$
$$=\mathrm{E}+\frac{2k^2s_1^2c_1^2\,(1+s_1)}{2s_1c_1d_1}$$

$$\text{Again, since } \mathrm{sn}\frac{2k}{3}=\frac{c_1}{d_1},\quad \frac{2s_1c_1d_1}{1-k^2s_1^4}=\frac{c_1}{d_1},$$

$$i.e. \qquad 1-2s_1+2k^2s_1^3-k^2s_1^4=0.$$

From the foregoing relation,

$$2\,k^2s_1^2c_1^2\,(1+s_1)$$
$$=2\,k^2s_1^2+2\,k^2s_1^3-2\,k^2s_1^4-2\,k^2s_1^5$$
$$=2\,k^2s_1^2+(k^2s_1^4+2\,s_1-1)-2\,k^2s_1^4$$
$$-2s_1\,(1-2s_1+2\,k^2s_1^3)$$
$$=-[1-2\,(2+k^2)s_1^2+5\,k^2s_1^4]$$

Hence relation (ii) is established.

To prove relation (iii).

$$\text{We have } 3\mathrm{E}\,(iu,\,k)=3i\left[\frac{\mathrm{sn}(u,k'(\mathrm{dn})u,k')}{\mathrm{sn}(u,k')}+u-\mathrm{E}(u,k')\right]$$

(*Vide* Dixon, *E. F. p.* 54.)

$$\text{write } u=\frac{\mathrm{K}'}{3};\ \mathrm{sn}\left(\frac{\mathrm{K}'}{3},k'\right)=\mathrm{S};\ \mathrm{cn}\left(\frac{\mathrm{K}'}{3};\,k'\right)=\mathrm{C} \text{ and } \mathrm{du}\frac{k'}{3}k'-0$$

$$\text{Hence } 3\mathrm{E}\left(\frac{i\mathrm{K}'}{3},k\right)=i\left[\frac{3\mathrm{SD}}{\mathrm{C}}+k'-3\,\mathrm{E}\left(\frac{\mathrm{K}'}{3},\,k'\right)\right]$$

Again from relation (iii),

$$2\,\mathrm{E}\left(\frac{\mathrm{K}'}{2'},k'\right)=\mathrm{E}'-\frac{1-2(2+k'^2)\mathrm{S}^2+5k'^2\mathrm{S}^4}{2\,\mathrm{SCD}}$$

Hence, after simplification,

$$3\,\mathrm{E}\left(\frac{i\mathrm{K}'}{3},k,\right) = i\,(\mathrm{K}'-\mathrm{E}') + \frac{i}{2\mathrm{SCD}}\,[1+2k^2\mathrm{S}'-k'^2\mathrm{S}^4]$$

$$= i(\mathrm{K}'-\mathrm{E}') - \frac{1}{2s_2c_2d_2}\,[c_2^4-2k^2s_2^2c_2^2-k'^2\mathrm{S}_2^4]$$

$$\left[\text{for } s_2 = sn\left(\frac{ik'}{3},\,k\right) = i\frac{\mathrm{S}}{\mathrm{C}};\; c_2 = \frac{1}{\mathrm{C}} \text{ and } d_2 = \frac{\mathrm{D}}{\mathrm{C}}\right]$$

$$= i\,(\mathrm{K}'-\mathrm{E}') - \frac{1-2\,(1+k^2)\,s_2^2+3\,k^2s_2^4}{2\,s_2c_2d_2}.$$

Hence from the relations (A), (i), (ii) and (iii) the proposed identity easily follows.

Question 966

(A. C. L. Wilkinson) :—Prove that

$$\mathrm{E}-3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = -\frac{k^2}{k'}\,\mathrm{cn}^2\left(\frac{\mathrm{K}}{3}\right)$$

Solution by F. H. V. Gulasekharam, M.A., and K. R. Rama Iyer,

In the solution to Q. 965 (*a*), it was shewn that

$$3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = \mathrm{E}+k^2\,\frac{s_1c_1(1+s_1)}{d_1},$$

where $s_1 = \mathrm{sn}\left(\frac{\mathrm{K}}{3}\right)$, $c_1 = \mathrm{cn}\left(\frac{\mathrm{K}}{3}\right)$ and $d_1 = \mathrm{dn}\left(\frac{\mathrm{K}}{3}\right)$.

It was also shewn that

$$1-2s_1+2k^2s_1^3-k^2s_1^4 = 0.$$

$$\therefore\quad (1-s_1)^2(1-k^2s_1^2) = (1-k^2)s_1^2$$

$$\therefore\quad (1-s_1)^2d_1^2 = k'^2s_1^2$$

$$\therefore\quad (1-s_1)d_1 = k's_1$$

$$\therefore\quad c_1^2d_1 = k's_1\,(1+s_1)$$

$$\therefore\quad 3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = \mathrm{E}+\frac{k^2}{k'}\,c_1^3$$

$$\therefore\quad \mathrm{E}-3\,\mathrm{E}\left(\frac{\mathrm{K}}{3}\right) = -\frac{k^2}{k'}\,c_1^3.$$

Additional Solution (1) *by F. H. V. Gulasekharam.* (2) *by the Proposer.*

(1) When $u_1+u_2+u_3+u_4=0$,

$$\sum_{r=1}^{4} \mathrm{E}u_r = \frac{k^2}{k^2-k'^2\,(c_1c_2c_3c_4)^{-1}} \sum_{r=1}^{4} \frac{s_r\,d_r}{c_r}.$$

where $s_r=\mathrm{sn}\ u_r$, $c_r=\mathrm{cn}\ u_r$ etc.

[*Vide.* J. I. M. S., Vol. X, pp. 338, 336.]

Write $u_4=-\mathrm{K}$, then when $u_1+u_2+u_3=\mathrm{K}$,

$$\sum_{r=1}^{3} \mathrm{E}u_r = \mathrm{E}+\frac{k^2}{k^1}c_1c_2c_3 \quad \ldots \quad \ldots \quad (1)$$

Now writing $u_1=u_2=u_3=\frac{\mathrm{K}}{3}$, we get the required result.

Note.—(*a*) By applying Jacobi's imaginary transformation to (1), we can prove that when $u_1+u_2+u_3=ik'$,

$$\sum_{r=1}^{3} \mathrm{E}u_r = i(\mathrm{K}'-\mathrm{E}')-\frac{ek'^2}{k.c_1c_2c_3}$$

$$+\sum_{r=1}^{3} \frac{s_r\,c_r}{d_r} \quad \ldots \quad (2)$$

Now, writing $u_1=u_2=u_3=\left(\frac{i\mathrm{K}'}{3}\right)$, we get a result for $\mathrm{E}\left(\frac{\mathrm{K}'}{3}\right)$.

(*b*) Again, we have when $u_1+u_2+u_3+u_4=0$,

$$\sum_{r+1}^{4} \mathrm{E}u_r = \frac{k^2}{1+k'^2(d_1d_2d_3d_4)^{-1}} \sum_{r=1}^{4} \frac{s_r\,c_r}{d_r}$$ (*Ibid*, pp. 338,336).

Write $u_4=-(\mathrm{K}+i\mathrm{K}')$

Hence when $u_1+u_2+u_3=\mathrm{K}+i\mathrm{K}'$,

$$\sum_{r=1}^{3} \mathrm{E}u_r = \mathrm{E}+i(\mathrm{K}'-\mathrm{E}')-\frac{i}{k'}\,d_1d_2d_3. \quad \ldots \quad (3)$$

Now writing $u_1=u_2=u_3=\left(\frac{\mathrm{K}+i\mathrm{K}'}{3}\right)$, we get a result for $\mathrm{E}\left(\frac{\mathrm{K}+i\mathrm{K}'}{3}\right)$.

(2) Consider the intersections of the curves

$$y^2=x(1-x)(1-k^2x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$y=(1-x)(ax+b) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The abscissae of the points of intersection are given by

$$F(x)=(1-x)(ax+b)^2-x(1-h^2x)=0, \quad \ldots \quad \ldots \quad (3)$$

and we have immediately $x_1x_2x_3=\frac{b^2}{a^2}$. $\ldots \quad \ldots \quad \ldots \quad (4)$

Applying Abel's Theorem to (3), we have

$$\frac{\partial F}{\partial x_1}\,dx_1=-2y_1(x_1da+db),$$

whence as in Abel's theorem

$$\frac{dx_1}{y_1}+\frac{dx_2}{y_2}+\frac{dx_3}{y_3}=0,$$

or, $u_1+u_2+u_3=$constant, where u_1, u_2, u_3 are the arguments of the three points of intersection.

From (2) $as_1^2+b=\frac{s_1d_1}{c_1}$, $as_2^2+b=\frac{s_2d_2}{c_2}$,

whence $$a=\frac{s_1d_1c_2-s_2d_1c_2}{c_1c_2(s_1^2s_2^2)}+\frac{du(u_1+u_2)}{c_1c_2\ \mathrm{sn}\ (u_1+u_2)},$$

$$b=\frac{s_1s_2\ \mathrm{cn}\ (u_1+u_2)}{c_1c_2\ \mathrm{sn}\ (u_1+u_2)};$$

combining these values with (4), we have

$$u_1+u_2+u_3=\mathrm{K} \text{ and } a=\frac{k'}{c_1c_2c_3}.$$

Consider now $$\sum\frac{x_1dx_1}{y_1}=-2\sum\frac{x_1^2da+x_1db}{\frac{\partial F}{\partial x_1}}=\frac{2}{a^2}da,$$

whence $$\Sigma k^2\ \mathrm{sn}^2u_1du_1=\frac{k^2da}{a^2} \text{ and since } \Sigma du_1=0.$$

We find on integrating and taking $u_1=u_2=u_3=\frac{k}{3}$ for the lower limit,

$$E(u_1)+E(u_2)+E(u_3)-3E\left(\frac{K}{3}\right)=\frac{k^2}{k'}\mathrm{cn}\ u_1\ \mathrm{cn}\ u_2\ \mathrm{cn}\ u_3-\frac{k^2}{k'}\ \mathrm{cn}^3\ \frac{K}{3},$$

where $u_1+u_2+u_3=\mathrm{K}$.

Writing $u_1=\mathrm{K}$, $u_2=u_3=0$, we have the given result.

Question 967.

(M. K. Kewalramani, M.A.) :—Prove that

$$1^{-5}+2^{-5}-4^{-5}-5^{-5}+7^{-5}+8^{-5}-\ldots=\frac{35\pi^5}{2916\sqrt{3}}$$

Solution by (1) *K. B. Madhava* (2) *by L. S. Vaidyanathan, M. A., and* (3) *by N. G. Leather.*

(1) Several problems of this nature have been solved in the pages of the J.I.M.S. (*vide* : Vol. 7.)

It may therefore be interesting to work out very general cases.

Write $$f(y)=\sum_{-\infty}^{\infty}\frac{e^{iny}}{x-n}$$

which satisfies the *formal* equation

$$(f'y)-ixf(y)=-i\sum_{\infty}e^{iny}$$

which, therefore, gives the *real* equation

$$[f'(y)-ixf(y)](1-e^{iy})=0.$$

Hence $f(y)=Ce^{ixy}$ where C is independent of y and y ranges from 0 to 2π.

Putting $y=\pi$, we have $C=2\pi i/(e^{2\pi ix}-I)$,
so that

$$\frac{2\pi ie^{ixy}}{e^{2\pi ix}-1}=\sum_{-\infty}^{\infty}\frac{e^{iny}}{x-n}(0<y<2\pi).$$

change y into $2\pi y$ and $2\pi ix$ into θ; we shall then have after some manipulation

$$(A)\qquad \frac{\theta e^{\theta y}}{e^{\theta}-1}=1+\sum_{\infty}{}'\frac{\theta e^{2n\pi iy}}{\theta-2n\pi i}0(<y<1$$

where the dash on the signa indicates the omission of $n=0$.

If we now remember, that the coefficient of $\frac{t^n}{n!}$ in $t\frac{e^{xt}-1}{e^t-1}$ is Bernoulli's Polynomial $\phi_n(x)$, and the coefficient of $\frac{t^n}{n!}$ in $\frac{t^2e^{xt}}{e^t-1}$, the Polynomial

$\phi'_n(x)$

we shall have

$$\phi'_{2m}(x)=2m\,\phi_{2m-1}\,\phi(x)\ (m>1)$$

and $$\phi'_{2m+1}(x)=(2m+1)[\phi_{2m}(x)+(-)^{m-1}B_m](m\geq 1)$$

where B_m in the *mth* Bernoullian number.

These results are given in § 94 of Bromwich: *Theory of Infinite Series*; vide: also J. I. M. S., Vol. 8., p. 148.

Expanding the right side of (A) and equating the coeffs. of

$$\frac{\theta^{2k}}{2k!}\ \text{and}\ \frac{\theta^{2k+1}}{(\theta k+1)!}$$

we have

$$\left.\begin{aligned} 2\sum_{n=1}^{\infty}\frac{\sin 2n\pi y}{(2n\pi)^{2r+1}} &= -\frac{1}{(2r+1)!}\phi_{2r+1}(y) \\ 2\sum_{n=1}^{\infty}\frac{\cos 2n\pi y}{(2n\pi)^{2r}} &= (-)^{r-1}\frac{1}{(2r)!}\phi_{2r}(y)+B_v \end{aligned}\right\} \quad \ldots \quad (B)$$

and other expressions.

The values of $\phi_r(y)$ up to $r=6$ are given in Bromwich, p. 236; the rest may be easily calculated.

From (B) we get a variety of results giving particular values to y and r. For the purposes of the present example $y=\frac{1}{6}$ and $r=2$. The result given by the Proposer seems to be incorrect; it ought to be.

$$\frac{17\pi^6}{2916\sqrt{3}}$$

(2) We know that

$$\tfrac{1}{2}(\pi-x)=\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots\ldots 0<x<2\pi$$

Integrating the above between π and x. we have

$$\frac{1}{2}\left(x\pi-\frac{x^2}{2}\right)-\frac{\pi^2}{6}=-\left[\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\right] \quad \ldots (1)$$

Since $$\sum\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Integrating (1) between the same limits thrice more we have the following equations.

$$\frac{1}{2}\left(\frac{x^2\pi}{2}-\frac{x^3}{6}\right)\frac{-\pi^2 x}{6}=-\left[\frac{\sin x}{1^3}+\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}+\ldots\right] \quad \ldots (2)$$

$$\frac{1}{2}\left(\frac{x^3\pi}{6}-\frac{x^4}{24}\right)-\frac{x^2\pi^2}{12}+\frac{\pi^4}{90}=\left[\frac{\cos x}{1^4}+\frac{\cos 2x}{2^4}+\frac{\cos 3x}{3^4}+\ldots\right] \ldots (3)$$

$$\frac{1}{2}\left(\frac{x^4\pi}{24}-\frac{x^5}{120}\right)-\frac{x^3\pi^2}{36}+\frac{\pi^4 x}{90}=\left[\frac{\sin x}{1^5}+\frac{\sin 2x}{2^5}+\frac{\sin 3x}{3^5}+\ldots\right] \quad (4]$$

putting $x=\frac{\pi}{3}$, we have

$$\frac{\pi^5}{3^5}\cdot\frac{17}{24}=\frac{\sqrt{3}}{4}.S$$

after simplifying, where S stands for the series whose sum we are seeking.

$$\therefore \quad S=\frac{\pi^5\,17}{3^5\times 12\times\sqrt{3}}=\frac{17\,\pi^5}{2916\sqrt{3}}.$$

(3) $\frac{\pi}{2n}\tan\frac{m\pi}{2n}=\frac{1}{n-m}-\frac{1}{n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}+\ldots$ (Hobson, § 294)

Differentiating with respect to m,

$$\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(3n-m)^2}+\frac{1}{(3n+m)^2}+\dots=\left(\frac{\pi}{2n}\right)^2\sec^2\frac{m\pi}{2n}.$$

Differentiating again

$$=\left[\frac{1}{(n-m)^3}-\frac{1}{(n+m)^3}+\frac{1}{(3n-m)^3}-\frac{1}{(3n+m)^3}+\dots\right]$$

$$=\left(\frac{\pi}{2n}\right)^3\sec^2\frac{m\pi}{2n}\tan\frac{m\pi}{2n}.$$

Differentiating again

$$3\left[\frac{1}{(n-m)^4}+\frac{1}{(n+m)^4}+\frac{1}{(3n-m)^4}+\frac{1}{(3n+m)^4}+\dots\right]$$

$$=\left(\frac{\pi}{2n}\right)^4\left[2\sec^2\frac{m\pi}{2n}\tan^2\frac{m\pi}{2n}+\sec^4\frac{m\pi}{2n}\right]$$

$$=\left(\frac{\pi}{2n}\right)^4\left[3\sec^4\frac{m\pi}{2n}-2\sec^2\frac{m\pi}{2n}\right].$$

Differentiating again

$$12\left[\frac{1}{(n-m)^5}-\frac{1}{(n+m)^5}+\frac{1}{(3n-m)^5}-\frac{1}{(3n+m)^5}\dots\right]$$

$$=\left(\frac{\pi}{2n}\right)^5\left[12\sec^4\frac{m\pi}{2n}-4\sec^2\frac{m\pi}{2n}\right]\tan\frac{m\pi}{2n}.$$

$$\therefore\quad \frac{1}{(n-m)^5}-\frac{1}{(n+m)^5}+\frac{1}{(3n-m)^5}-\frac{1}{(3n+m)^5}-\dots\dots$$

$$=\left(\frac{\pi}{2n}\right)^5\left[\sec^4\frac{m\pi}{2n}-\frac{1}{3}\sec^2\frac{m\pi}{2n}\right]\tan\frac{m\pi}{2n}.$$

Put $n=3,\ m=2$

$$\frac{1}{1^5}-\frac{1}{5^5}+\frac{1}{7^5}-\frac{1}{11^5}+\dots=\left(\frac{\pi}{6}\right)^5\left[16-\frac{4}{3}\right]\sqrt{3}$$

$$=\frac{\pi^5.\ 44}{6^5.\sqrt{3}}=\frac{\pi^5.\ 11}{2^3.\ 1^5.\sqrt{8}}.$$

Put $n=3,\ m=1$

$$\frac{1}{2^5}-\frac{1}{4^5}+\frac{1}{8^5}-\frac{1}{10^5}+\dots=\left(\frac{\pi}{6}\right)^5\left[\frac{16}{9}-\frac{4}{9}\right]\frac{1}{\sqrt{3}}$$

$$=\frac{\pi^5.\ 4}{6^5.\ 3\sqrt{3}}=\frac{\pi^5}{2^3.\ 3^5.\sqrt{3}}.$$

Adding, we get

$$\frac{1}{1^5}+\frac{1}{2^5}-\frac{1}{4^5}-\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{8^5}-\frac{1}{10^5}-\frac{1}{11^5}-\dots$$

$$=\frac{\pi^5\sqrt{[33+1]}}{2^3.\ 3^5.\sqrt{3}}=\frac{17\pi^5}{2916\sqrt{3}}.$$

QUESTIONS FOR SOLUTION.

1027. (SELECTED) :—If p is a prime number and equal to $2n+1$, show that

$$(2n)!\equiv(-1)^n 2^{4n}.(n!)^2, \text{ (mod, } p^2)$$

1028. (N. P. PANDYA) :—Two tangents to a parabola a are fixed, and a third one is variable. Find the locus of the centroid of a triangle inscribed in the parabola, having its sides parallel to the three tangents.

1029. (SELECTED) :—Prove that

$$\left(\frac{d}{dx}e^x\right)^n uv =$$

$$\sum_{r=0}^{n} \frac{n\,1}{r\,!\,(n-r)\,!}\left[\left(\frac{d}{dx}e^x\right)^r e^{-rx}u\right]\left[\left(\frac{d}{dx}e^x\right)^{n-r}\left(\frac{d}{dx}e^x\right).{}^{n-r}e^{rx}v\right].$$

1030. (SELECTED) :—If $x^{2n-1}+y^{2n-1}+a_1{}^{2n-1}+a_2{}^{2n-1}+\ldots a_{2r}{}^{2n-1}=0$, for integral values of n from 1 to r inclusive, then

$$(x+a_1)(x+a_2)\ldots(x+a_{2r})=(y+a_1)(y+a_2)\ldots(y+a_{2r}).$$

1031. (C. KRISHNAMACHARI) :—With the usual notation in elliptic functions, show that if $u+v+w=0$,

$$p(u)p(v)p(w)=\tfrac{1}{4}g_3+\tfrac{1}{4}\left\{\frac{p'(u)p(v)-p'(v)p(u)}{p(u)-p(v)}\right\}^2$$

Deduce that

$$p(z)\{p(z+\omega_1)+p(z+\omega_2)+p(z+\omega_3)\}+\tfrac{1}{4}g_2$$
$$=\tfrac{1}{4}p'(z)\left\{\frac{1}{e_1-p(z)}+\frac{1}{e_2-p(z)}+\frac{1}{e_3-p(z)}\right\}$$

Note.—where $p(u)$ denotes the Weierstrassian function.

1032. (N. P. PANDYA) :—The month of April, 1919, coincides with the month of Chaitra of the Samvat year 1975. Find other pairs of coincident months in the next 300 years.

1033. (V. RAMASWAMI AIYAR, M.A.) :—If two figures in a plane, one consisting of lines, and the other, of points, be in ortho-polar correspondence, (that is with respect to some triangle of reference ABC) prove that they will continue to be in ortho-polar correspondence under any relative motion of translation.

Show also that the circum circle of the triangle of reference continues invariable in the figure of lines and the nine-points circle, in the figure of points.

1034. (V. Ramaswami Aiyar, M.A.) :–Prove that the locus of the double foci of a system of confocal bicircular quarties consists of the two circular cubics which form the extreme members of the system.

1035. (K. Ananda Rao):––Suppose that

$$f(x) = a_o + a_1 x + \ldots + a_n x^n$$

is a polynomical with integral (positive, negative or zero) co-efficients, and that is *irreducible* that is to say it cannot be expressed as the product of two polynomials whose co-efficients are also integral. Then show that $f(x)$ cannot have a root of multiplicity greater than $\frac{1}{2}\, n$?.

1036. (A. C. L. Wilkinson).—A variable circle touches an ellipse at a fixed point. Show that the locus of the point of intersection of the other two common tangents of the circle and ellipse is the confocal hyperbola passing through the fixed point.

1037. (A. C. L. Wilkinson) :—If one angle of a triangle is 60°, prove that the Euler line of the triangle is equally inclined to the two sides containing the angle of 60°.

1038. (A. C. L. Wilkinson) :—The equation of a conic referred to tangential areal coordinates is

$$al^2 + bm^2 + cn^2 + 2hlm + 2gnl + 2fmn = o,$$

show that the magnitudes of its semi-axes are given by the equation

$$\mathrm{I}^3 x^4 + 4\,\mathrm{I}\,\mathrm{J}\,\mathrm{R}^2 x^2 + 16\,\mathrm{R}^4 \Delta \sin^2 \mathrm{A} \sin^2 \mathrm{B} \sin^2 \mathrm{C} = 0,$$

where R is the radius of the circumcircle of ABC, the triangle of reference, and

$$\mathrm{I} = a + b + c + 2f + 2g + 2h$$

$$\mathrm{J} = \mathrm{A} \sin^2 \mathrm{A} + \mathrm{B} \sin^2 \mathrm{B} + \mathrm{C} \sin^2 \mathrm{C} - 2\,\mathrm{F} \cos \mathrm{A} \sin \mathrm{B} \sin \mathrm{C} - 2\mathrm{G} \cos \mathrm{B} \sin \mathrm{C} \sin \mathrm{A} - 2\,\mathrm{H} \cos \mathrm{C} \sin \mathrm{A} \sin \mathrm{B}.$$

$$\mathrm{S} = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \text{ and } \mathrm{A} = \frac{\partial \Delta}{\partial a}, \quad \mathrm{F} = \tfrac{1}{2} \frac{\partial \Delta}{\partial f} \text{ etc.,}$$

Hence, or otherwise, prove that if a conic is inscribed in a triangle, the product of the squares of its semiaxes is $4\,\mathrm{R}\alpha\beta\gamma$, where R is the radius of the circumcircle of the triangle and α, β, γ are the perpendiculars from the centre of the conic on the lines joining the middle points of the sides of the triangle.

1039. (A. NARASINGA RAO) :—The integer 142857 is such that every cyclic permutation of its digits is a multipleof itself. Find other numbers of the same type.

1040. (A. NARASINGA RAO) :—Show that any small displacement of a rigid body may be resolved into 6 rotations about the edges of an arbitrary tetrahedron A B C D. If the motion be one of pure rotation about an instantaneous axis, prove that

$$^{w}ab.\ ^{w}cd + ^{w}bc.\ ^{w}ad + ^{w}ca.\ ^{w}bd = 0,$$

^{w}ab being angular velocity about the edge A B, with a similar notation for the others.

Determine the angular velocity about the axis ; also find the relation between the w's when the motion is one of pure translation.

1041. (A. NARASINGA RAO) :—Prove the following analogue of Ceva's Theorem in three dimensions.

A, B, C, D, are 4 points on the faces of a tetrahedron A B C D. If AA_1 BB_1 CC_1 & DD_1 are the generators of a hyperboloid, we have four relations of the type

$$\frac{\triangle A_1CD}{\triangle B_1CD}\cdot\frac{\triangle B_1AD}{\triangle C_1AD},\ \frac{\triangle C_1BD}{\triangle A_1BD} = 1.$$

Are these conditions all independant? If not, determine the relation between them.

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MR. V. RAMASWAMI AIYAR, M.A.

Some Definite Integrals.

By S. Ramanujan.

I have shown elsewhere * that the definite integrals

$$\phi_w(t)=\int_0^\infty \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi w x^2} dx,$$

$$\psi_w(t)=\int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi w x^2} dx$$

can be evaluated in finite terms if w is any rational multiple of i.

In this paper I shall show, by a much simpler method, that these integrals can be evaluated not only for these values but also for many other values of t and w.

Now we have

$$\phi_w(t)=2\int_0^\infty \int_0^\infty \frac{\cos 2\pi x z}{\cosh \pi z} \cos \pi t x . e^{-\pi w x^2} dx\, dz$$

$$=\frac{e^{-\frac{\pi t^2 w'}{4}}}{\sqrt{w}} \int_0^\infty \frac{\cosh \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx$$

where w' stands for $1/w$.

It follows that

$$\phi_w(t)=\frac{1}{\sqrt{w}} e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(i t w') \qquad (1)$$

Again

$$\phi_w(t+w)=\frac{1}{\sqrt{w}} e^{-\frac{\pi (t+w)^2}{4} w'} \times$$

$$\int_0^\infty \frac{\cosh \frac{\pi t}{w} x \cosh \pi x+\sinh \frac{\pi t}{w} x \sinh \pi x}{\cosh \pi x} e^{-\frac{\pi}{w} x^2} dx$$

$$=\frac{1}{\sqrt{w}} e^{-\frac{\pi (t+w)^2}{4w}} \left\{ \tfrac{1}{2}\sqrt{w} e^{\pi t^2/4w} + 2\int_0^\infty \int_0^\infty \frac{\sin 2\pi x z}{\sinh \pi z} \sinh \frac{\pi t x}{w} e^{-\frac{\pi}{w} x^2} dx\, dz \right\}$$

* *The Messenger of Mathematics*, New Series, No. 522, Oct. 1914.

$$= \frac{1}{\sqrt{w}} e^{-\frac{\pi(t+w)^2}{4w}} \left\{ \tfrac{1}{2}\sqrt{w}\, e^{\pi t^2/4w} + \right.$$

$$\left. \sqrt{w} e^{\pi t^2/4w} \int_0^\infty \frac{\sin \pi t x}{\sinh \pi x} e^{-\pi w x^2} dx. \right.$$

In other words

$$e^{\pi t^2/4w} \{ \tfrac{1}{2} + \psi_w(t) \} = e^{\frac{\pi(t+w)^2}{4w}} \phi_w(t+w). \tag{2}$$

It is obvions that

$$\begin{cases} \phi_w(t) = \phi_w(-t) \\ \psi_w(t) = -\psi_w(-t) \end{cases} \tag{3}$$

From (1), (2) and (3) we easily find that

$$\tfrac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\frac{\pi t^2}{4w}} \left\{ \tfrac{1}{2} - \psi_{w'}\left(\frac{it}{w} + i\right) \right\} \tag{4}$$

It is easy to see that

$$\phi_w(i) = \frac{1}{2\sqrt{w}}; \ \psi_w(i) = \frac{i}{2\sqrt{w}}; \ \phi_w(w) = \tfrac{1}{2} e^{-\frac{\pi w}{4}};$$

$$\tfrac{1}{2} - \psi_w(w) = e^{-\frac{\pi w}{4}} \phi_w(0); \ \phi_w(w \pm i) = \left(\frac{1}{2\sqrt{w}} \pm \frac{i}{2}\right) e^{-\frac{\pi w}{4}};$$

$$\psi_w(w \pm i) = \tfrac{1}{2} \pm \frac{i}{2\sqrt{w}} e^{-\frac{\pi w}{4}}; \ \phi_w(\tfrac{1}{2}w) + \psi_w(\tfrac{1}{2}w) = \tfrac{1}{2}.$$

Again we see that

$$\phi_w(t+i) - \phi_w(t-i) = \frac{1}{\sqrt{w}} e^{-\frac{\pi t^2}{4w}}; \tag{5}$$

and

$$\psi_w(t+i) - \psi_w(t-i) = \frac{i}{\sqrt{w}} e^{-\frac{\pi t^2}{4w}}. \tag{6}$$

From (1) and (5) we deduce that

$$e^{\frac{\pi(t+w)^2}{4w}} \phi_w(t+w) + e^{\frac{\pi(t-w)^2}{4w}} \phi_w(t-w) = e^{\frac{\pi t^2}{4w}}. \tag{7}$$

Similarly from (4) and (6) we obtain

$$e^{\frac{\pi(t+w)^2}{4w}} \left\{ \frac{1}{2} - \psi_w(t+w) \right\} = e^{\frac{\pi(t-w)^2}{4w}} \left\{ \frac{1}{2} + \psi_w(t-w) \right\}. \tag{8}$$

It is easy to deduce from (5) that if n is a positive integer, then

$$\phi_w(t)+(-1)^{n+1}\phi_w(t\pm 2ni) \tag{9}$$

$$=\frac{1}{\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}-e^{-\frac{\pi}{4w}(t\pm 3i)^2}+e^{-\frac{\pi}{4w}(t\pm 5i)^2}\right.$$

$$\left.-\ldots\text{to } n \text{ terms}\right\}.$$

Similarly from (6) we have

$$\psi_w(t)-\psi_w(t\pm 2ni) \tag{10}$$

$$=-\frac{1}{\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t+i)^2}+e^{-\frac{\pi}{4w}(t+3i)^2}+e^{-\frac{\pi}{4w}(t+5i)^2}\right.$$

$$\left.-\ldots\text{to } n \text{ terms}\right\}$$

Again from (7) we have

$$e^{\frac{\pi t^2}{4w}}\phi_w(t)+(-1)^{n+1}e^{\frac{\pi(t+2nw)^2}{4w}}\phi_w(t+2nw)$$

$$=e^{\frac{\pi}{4w}(t+w)^2}-e^{\frac{\pi}{4w}(t+3w)^2}+e^{\frac{\pi}{4w}(t+5w)^2} \tag{11}$$

$$-\ldots\text{to } n \text{ terms};$$

and from (9)

$$e^{\frac{\pi t^2}{4w}}\left\{\frac{1}{2}+\psi_w(t)\right\}+(-1)^{n+1}e^{\frac{\pi(t+2nw)^2}{4w}}\times \tag{12}$$

$$\left\{\frac{1}{2}+\psi_w(t+2nw)\right\}$$

$$=e^{\frac{\pi}{4w}(t+2w)^2}-e^{\frac{\pi}{4w}(t+4w)^2}+e^{\frac{\pi}{4w}(t+6w)^2}$$

$$-\ldots\text{to } n \text{ terms.}$$

Now combining (9) and (11) we deduce that if m and n are positive integers and $s=t+2mw\pm 2ni$, then

$$\phi_w(s)+(-1)^{(m+1)(n+1)}e^{-\frac{1}{2}\pi m(s+t)}\phi_w(t) \tag{13}$$

$$=e^{-\frac{\pi s^2}{4w}}\left\{e^{\frac{\pi}{4w}(s-w)^2}-e^{\frac{\pi}{4w}(s-3w)^2}+e^{\frac{\pi}{4w}(s-5w)^2}\right.$$

$$\left.\ldots\text{ to } m \text{ terms}\right\}$$

$$+\frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}\right.$$

$$-e^{-\frac{\pi}{4w}(t\pm 3i)^2}+e^{-\frac{\pi}{4w}(t\pm 5i)^2}-\ldots\text{to } n \text{ terms}\left.\right\}.$$

Similarly combining (10) and (12) we obtain

$$\frac{1}{2}-\psi_w(s)+(-1)^{mn+m+1}\,e^{-\frac{1}{2}\pi m(s+t)}\left\{\frac{1}{2}-\psi_w(t)\right\} \tag{14}$$

$$=e^{-\frac{\pi s^2}{4w}}\left\{e^{\frac{\pi}{4w}(s-2w)^2}-e^{\frac{\pi}{4w}(s-4w)^2}\right.$$

$$\left.+e^{\frac{\pi}{4w}(s-6w)^2}-\ldots\text{to } m \text{ terms}\right\}$$

$$\pm(-1)^{mn+m+1}\frac{i}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}\right.$$

$$\left.+e^{-\frac{\pi}{4w}(t\pm 3i)^2}+e^{-\frac{\pi}{4w}(t\pm 5i)^2}+\ldots\text{to } n \text{ terms}\right\}$$

where s and t have the same relation as in (13).

Suppose now that $s=t$ in (13) and (14). Then we see that

if $w=\frac{in}{m}$, then

$$\phi_w(t)\left\{1+(-1)^{(m+1)(n+1)}e^{-\pi mt}\right\} \tag{15}$$

$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-w)^2}-e^{\frac{\pi}{4w}(t-3w)^2}\right.$$

$$\left.+e^{\frac{\pi}{4w}(t-5w)^2}-\ldots\text{to } m \text{ terms}\right\}$$

$$+\frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{\pi}{4w}(t-i)^2}\right.$$

$$\left.-e^{-\frac{\pi}{4w}(t-3i)^2}+\ldots\text{to } n \text{ terms}\right\};$$

$$\left\{\frac{1}{2}-\psi_w(t)\right\}\left\{1+(-1)^{mn+m+1}e^{-\pi mt}\right\} \tag{16}$$

$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-2w)^2}-e^{\frac{\pi}{4w}(t-4w)^2}+\ldots\text{to } m \text{ terms}\right\}$$

$$+(-1)^{mn+m}\frac{i}{\sqrt{w}}e^{-\pi mt}\left\{e^{-\frac{\pi}{4w}(t-i)^2}\right.$$

$$\left.+e^{-\frac{\pi}{4w}(t-3i)^2}+\ldots\text{to } n \text{ terms}\right\}$$

where $\sqrt{w}$ should be taken as

$$e^{\frac{i\pi}{4}}\sqrt{\left(\frac{n}{m}\right)}.$$

In (15) and (16) there is no loss of generality in supposing that one of the two numbers m and n is odd.

Now equating the real and imaginary parts in (15), we deduce that if m and n are positive integers of which one is odd, then

$$2\cosh nt\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\cos\left(\frac{\pi m x^2}{n}\right)dx \tag{17}$$

$$=[\cosh\{(1-n)t\}\cos(\pi m/4n)-\cosh\{(3-n)t\}\cos(9\pi m/4n)$$

$$+\sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.$$

$$\left.-\cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\cos\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{4m}\right)+\ldots\text{to } m \text{ terms}\right];$$

and

$$2\cosh nt\int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\sin\left(\frac{\pi m x^2}{n}\right)dx \tag{18}$$

$$=-[\cosh\{(1-n)t\}\sin(\pi m/4n)$$

$$-\cosh\{(3-n)t\}\sin(9\pi m/4n)$$

$$+\cosh\{(5-n)t\}\sin(25\pi/4n)-\ldots\text{to } n \text{ terms}]$$

$$+\sqrt{\left(\frac{n}{m}\right)}\left[\cosh\left\{\left(1-\frac{1}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{\pi n}{4m}\right)\right.$$

$$\left.-\cosh\left\{\left(1-\frac{3}{m}\right)nt\right\}\sin\left(\frac{\pi}{4}-\frac{nt^2}{\pi m}+\frac{9\pi n}{\pi m}\right)\right.$$

$$\left.+\ldots\text{to } n \text{ terms}\right].$$

Equating the real and imaginary parts in (16), we can find similar expressions for the integrals

$$\int_0^\infty \frac{\sin tx}{\sinh \pi x}\sin\left(\frac{\pi m x^2}{n}\right)dx,\quad \int_0^\infty \frac{\sin tx}{\sinh \pi x}\cos\left(\frac{\pi m x^2}{n}\right)dx.$$

From these formulae we can evaluate a number of definite integrals, such as

$$\int_0^\infty \frac{\cos 2\pi tx}{\cosh \pi x}\cos \pi x^2\,dx=\frac{1+\sqrt{2}\sin \pi t^2}{2\sqrt{2}\cosh \pi t},$$

$$\int_0^\infty \frac{\cos 2\pi tx}{\cosh \pi x}\sin \pi x^2\,dx=\frac{-1+\sqrt{2}\cos \pi t^2}{2\sqrt{2}\cosh \pi t},$$

$$\int_0^\infty \frac{\sin 2\pi tx}{\sinh \pi x}\cos \pi x^2\,dx=\frac{\cosh \pi t-\cos \pi t^2}{2\sinh \pi t},$$

$$\int_0^\infty \frac{\sin 2\pi tx}{\sinh \pi x}\sin \pi x^2\,dx=\frac{\sin \pi t^2}{2\sinh \pi t},$$

and so on.

Again supposing that $s=-t$ in (13), we deduce that if $t=m\,w\pm n\,i$ where m and n are positive integers of which one at least is odd, then

$$\phi_w(t)=\frac{1}{2}e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-w)^2}-e^{\frac{\pi}{4w}(t-3w)^2}+\ldots\text{to } m \text{ terms}\right\}$$
$$+\frac{1}{2\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t\pm i)^2}-e^{-\frac{\pi}{4w}(t\pm 3i)^2}+\ldots\text{to } n \text{ terms}\right\}. \qquad (19)$$

This formula is not true when both m and n are even.

If $t=m\,w\pm n\,i$ where m and n are both even, then

$$\phi_w(t)+(-1)^{(1+\frac{1}{2}m)\,(1+\frac{1}{2}n)}\,e^{-\frac{1}{4}\pi m t}\,\phi_w(0) \qquad (20)$$
$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-w)^2}-e^{\frac{\pi}{4w}(t-3w)^2}+\ldots\text{to } \tfrac{1}{2}\,m \text{ terms}\right\}$$
$$+\frac{(-1)^{(1+\frac{1}{2}m)\,(1+\frac{1}{2}n)}}{\sqrt{w}}\,e^{-\frac{1}{4}\pi m\,t}$$
$$\left\{e^{\frac{\pi}{4w}}-e^{\frac{9\pi}{4w}}+e^{\frac{25\pi}{4w}}-\ldots\text{to } \tfrac{1}{2}\,n \text{ terms}\right\}.$$

This is easily obtained by putting $t=0$ and then changing s to t in (13). Similarly from (14) we deduce that if $t=m\,w\pm n\,i$ where m and n are both even, or both odd, or m is even and n is odd, then

$$\psi_w(t)=-\frac{1}{2}e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-2w)^2}-e^{\frac{\pi}{4w}(t-4w)^2}\right. \qquad (21)$$
$$\left.+\ldots\text{to } \tfrac{1}{2}m \text{ terms}\right\}$$
$$\pm\frac{i}{2\sqrt{w}}\left\{e^{-\frac{\pi}{4w}(t\mp i)^2}+e^{-\frac{\pi}{4w}(t\mp 3i)^2}+\ldots\text{to } n \text{ terms}\right\}.$$

If $t=mw\pm ni$ where m is odd and n is even, then

$$\tfrac{1}{2}-\psi_w(t)+\{(m-1)^{1+\frac{1}{4}(m-1)(n+2)} \qquad (22)$$
$$e^{-\frac{1}{4}\pi\{(m-1)t+mw\}}\,\phi_w(o)$$
$$=e^{-\frac{\pi t^2}{4w}}\left\{e^{\frac{\pi}{4w}(t-2w)^2}-e^{\frac{\pi}{4w}(t-4w)^2}\right.$$
$$\left.\ldots+\text{to } \tfrac{1}{2}\,(m-1) \text{ terms}\right\}$$

$$\pm(-1)^{1+\frac{1}{4}(m-1)(n+2)}\frac{i}{\sqrt{w}}e^{-\frac{1}{4}\pi(m-1)(t+w)}$$

$$\left\{e^{-\frac{\pi}{4w}(w\mp i)^2}-e^{-\frac{\pi}{4w}(w\mp 3i)^2}+\ldots\text{to }\tfrac{1}{2}n\text{ terms.}\right\}$$

This is obtained by putting $t=w$ in (14). A number of definite integrals such as the following can be evaluated with the help of the above formulae.

$$\int_0^\infty \frac{\cos\pi tx}{\cosh\pi x}e^{-\pi(t+i)x^2}dx=\frac{1+i}{2\sqrt{2}}e^{-\frac{\pi t}{4}}\left\{1-\frac{i}{\sqrt{(t+i)}}\right\},$$

$$\int_0^\infty \frac{\sin\pi tx}{\sinh\pi x}e^{-\pi(t+i)x^2}dx=\frac{1}{2}-\frac{1+i}{2\sqrt{2}}\cdot\frac{e^{-\frac{\pi t}{4}}}{\sqrt{(t+i)}},$$

and so on.

Fractional Differentiation.*

By M. T. Naraniengar.

[This paper relates to a method of fractional differentiation which is apparently a generalization of the customary processes of differentiation of the Calculus. The idea here developed is akin to that of expressing factorials of fractional numbers by gamma functions and the results arrived at are not altogether uninteresting. So far as the author is aware, no reference to the ideas of this paper is to be found in any previous paper in the English language. In other languages also, such references are few and are of a fragmentary nature. M Paul Dienes in his "*Leçons sur les singularities des functions analytiques*" (one of the collections of Borel's pamphlets) draws casual attention to the genralized derivate of Riemann and of the analogous process of integration of a fractional or incommensurable index by Hadamard. The reference to the work of Riemann is to "*Versuch allg. Auffasung der Integ. and Differ.*" (1847) and "*Mathematische Werke*;" but the articles by Voss and Pringsheim in the *Encyclopaedie der Math Wissenschaften* do not make any mention of fractional differentiation. Forsyth makes a casual reference to the operator $\left(\frac{d}{dt}\right)^{\frac{1}{2}}$ at p. 393, (*Diffl. Eqns.*, 1885,) but abstains from developing any properties of the operator.

The International Catalogue of Scientific Literature gives the following paper, which may have some bearing on the subject matter; but the present writer regrets he has not been able to get access to it,

R. E. Moritzer: "Generalization of the Differentiation Process," *Amer. Jour. of Math.* 24 (1902) 257—302.]

(i) Ordinary Differentiation.

1. The laws of combination of the operator D or $\frac{d}{dx}$ are extended by considering fractional indices in the index law as expressed by the equation

$$D^p D^q u = D^{p+q} u. \tag{1}$$

Thus we may write

$$D^{\frac{1}{2}} D^{\frac{1}{2}} u = Du, \tag{2}$$

$$D^{\frac{1}{3}} D^{\frac{1}{3}} D^{\frac{1}{3}} u = Du, \tag{3}$$

and so on. The meaning of $D^{\frac{1}{2}}$ or $D^{\frac{1}{3}}$ in the above has therefore to be inferred from these functional equations.

* Presented to the Indian Mathematical Conference, Bombay, January 1919.

Let us consider the function x^n. We know that the operator D as applied to it signifies that the function has to be multiplied by the index n and unity subtracted from the index. Hence the symbol $D^{\frac{1}{2}}$ should mean an operation which when repeated twice in succession will be equivalent to the above. In other words, if $D^{\frac{1}{2}}$ is equivalent to multiplication by R (n) and diminution of the index by $\frac{1}{2}$,

i.e., if $$D^{\frac{1}{2}}(x^n) = R(n)x^{n-\frac{1}{2}} \quad (4)$$

then $$D^{\frac{1}{2}}D^{\frac{1}{2}}(x^n) = R(n)R(n-\tfrac{1}{2})x^{n-1}\ ; \quad (5)$$

and therefore $$R(n)R(n-\tfrac{1}{2}) = n, \quad (6)$$

which is the functional equation defining the new symbol R (n).

2. To solve the above equation, we may proceed thus :—

We have $$R(n).\ R(n-\tfrac{1}{2}) = n,$$
$$R(n-\tfrac{1}{2}).\ R(n-1) = n-\tfrac{1}{2}.$$
$$\therefore \quad R(n)/R(n-1) = n/(n-\tfrac{1}{2}).$$

But $$\Gamma(n+1) = n\Gamma(n),$$

and $$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\ \Gamma(n-\tfrac{1}{2})\ ;$$

so that $$\frac{R(n)}{R(n-1)} = \frac{\Gamma(n+1)\ \Gamma(n-\frac{1}{2})}{\Gamma(n)\ \Gamma(n+\frac{1}{2})}$$
$$= \frac{\Gamma(n+1)/\Gamma(n+\frac{1}{2})}{\Gamma(n)/\Gamma(n-\frac{1}{2})}\ .$$

Hence $$R(n) \propto \Gamma(n+1)/\Gamma(n+\tfrac{1}{2})$$
$$= \frac{\lambda\ \Gamma(n+1)}{\Gamma(n+\frac{1}{2})}, \text{ say.}$$

Then $$R(n-\tfrac{1}{2}) = \lambda\Gamma(n+\tfrac{1}{2})\Gamma(n)$$

$$\therefore \quad R(n).\ R(n-\tfrac{1}{2}) = \lambda^2\ \Gamma(n+1)/\Gamma(n)$$
$$= \lambda^2\ n = n, \text{ by (6)}$$

Thus $$\lambda^2 = 1 \quad \text{or} \quad \lambda = 1|\ ;$$

so that $$R(n) = \Gamma(n+1)/\Gamma(n+\tfrac{1}{2}), \quad \ldots \quad (7)$$

and $$\frac{1}{R(n)} = \frac{1}{\sqrt{\pi}}\left\{\frac{1}{n+\frac{1}{2}} + \frac{\frac{1}{2}}{n+\frac{3}{2}} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{(n+\frac{5}{2})^3_1} + \ldots\right.$$

[Bromwich : *Inf. Series*, p. 477, Ex. 51.]

3. **Next consider $D^{\frac{1}{3}}(x^n)$.**

Proceeding as before, we have

$$R(n)\ R(n-\tfrac{1}{3})\ R(n-\tfrac{2}{3}) = n,$$

and $$R(n-\tfrac{1}{3})\ R(n-\tfrac{2}{3})\ R(n-1) = n-\tfrac{1}{3}.$$

$$\therefore \qquad R(n)/R(n-1) = n/(n-\tfrac{1}{3})$$
$$= \frac{\Gamma(n+1)\,\Gamma(n-\frac{1}{3})}{\Gamma(n)\,\Gamma(n+\frac{2}{3})}$$
$$= \frac{\Gamma(n+1)/\Gamma(n+\frac{2}{3})}{\Gamma(n)/\Gamma(n-\frac{1}{3})}.$$

$$\therefore \qquad R(n) = \lambda\,\Gamma(n+1)/\Gamma(n+\tfrac{2}{3}).$$

It is easily proved that $\lambda = 1$ as in § 2, and we have finally

$$R(n) = \Gamma(n+1)/\Gamma(n+\tfrac{2}{3}),$$

or
$$\frac{1}{R(n)} = \frac{1}{\Gamma(\frac{1}{3})}\left\{\frac{1}{n+\frac{2}{3}} + \frac{2}{3(n+\frac{5}{3})} + \frac{2\cdot5}{3\cdot6(n+\frac{8}{3})} + \cdots\right\}$$

4. Generally $D^{\frac{1}{p}}(x^n) = R(n)\,x^{n-\frac{1}{p}}$,

where
$$R(n) = \Gamma(n+1)/\Gamma(n+\frac{p-1}{p})$$

and
$$\frac{1}{R(n)} = \frac{1}{\left(\Gamma\frac{1}{p}\right)}\left\{\frac{1}{(n+1-\frac{1}{p})} + \frac{p-1}{p\left(n+2-\frac{1}{p}\right)} + \right.$$
$$\left. \frac{(p-1)(2p-1)}{p\cdot 2p(n+3-\frac{1}{p})} + \cdots\right\}$$

5. The negative factional index is readily interpreted as follows :—

$$D^{\frac{1}{2}}(x^n) = x^{n-\frac{1}{2}}\Gamma(n+1)/\Gamma(n+\tfrac{1}{2}),$$

since
$$D^{-\frac{1}{2}}(x^{n-\frac{1}{2}}) = x^n\Gamma(n+\tfrac{1}{2})/\Gamma(n+1),$$

or
$$D^{-\frac{1}{2}}(x^n) = x^{n+\frac{1}{2}}\Gamma(n+1)/\Gamma(n+\tfrac{3}{2});$$

and so on.

6. In the following we shall use the notation $x!$ instead of $\Gamma(x+1)$.

Theorem I. *The generalised Leibnitz's Theorem.*

If u, v be any two polynomials in x, then

$$D^p(uv) = (D_1 + D_2)^p uv$$

whatever p may be, integral or fractional, positive or negative.

The proof of the theorem depends upon the well known property of the hypergeometric function given in Forsyth.

Thus

$$(D_1+D_2)^p(x^m x^n) = D_1^p(x^m) + \frac{p}{1!} D_1^{p-1}(x^m) D_2(x^n) + \dots$$

$$= \frac{m!}{(m-p)!} x^{m-p} x^n + \frac{p}{1!} \frac{m!}{(m-p+1)!} \frac{n!}{(n-1)!} x^{m-p+1} x^{n-1} + \dots$$

$$= \frac{m!}{(m-p)!} x^{m+n-p} \left\{ 1 + \frac{p.n}{m-p+1} + \frac{p(p-1)}{2!} \frac{n(n-1)}{(m-p+1)(m-p+2)} + \dots \right\}$$

$$= \frac{m!}{(m-p)!} x^{m+n-p} F_1 \left\{ -p, -n, \; m-p+1 \right\}$$

$$= x^{m+n-p} \frac{(m+n)!}{(m+n-p)!}, \quad \left[\text{Forsyth, § 126} \right]$$

which is the same as $D^p(x^m x^n)$

Hence writing $(u.v)$ in the form $\Sigma A(x^m . x^n)$ the result follows.

The following further extension is obvious :—

$$D^p(u,v.w.\dots\dots) = (D_1+D_2+D_3+\dots)^p \{ u.v.w\dots \}$$

6. **Theorem II.** *Let* $f(x) = \Sigma (a_n x^n)$ *and* $F(x) = \Sigma (b_n x^n)$ *where* n *may be integral or fractional ; then the constant term in* $f(D)\ F(x)$ *is equal to the constant term in* $F(D)\ f(x)$.

For,
$$f(D)\ F(x) = \Sigma(a_p D^p).\ \Sigma (b_q x^q)$$
$$= \Sigma\, a_q b_q \frac{q!}{(q-p)!} x^{q-p}.$$

Hence the constant term is $\Sigma (a_n b_n . n!)$ and its symmetry shows that it is the same as the constant term in $F(D)\ f(x)$.

8. *The generalised Binomial Series.*

Consider
$$z = n_{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + n_{\frac{3}{2}} \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \dots$$

where n_r denotes $\dfrac{n!}{(n-r)!}$.

Then
$$D(z) = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{(\frac{1}{2})!} + n_{\frac{3}{2}} \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + \dots$$

But
$$n_{\frac{3}{2}} = \frac{n!}{(n-\frac{3}{2})!} = \frac{n!}{(n-\frac{1}{2})!} . (n-\tfrac{1}{2}) = n_{\frac{1}{2}} (n-\tfrac{1}{2}), \text{ &c., &c.}$$

$$D(z) = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} + n_{\frac{1}{2}} z - \left\{ \tfrac{1}{2} n_{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + \tfrac{3}{2} n_{\frac{3}{2}} \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \dots\dots \right\}$$

$$\therefore \qquad \mathrm{D}(z) = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} + nz - x\mathrm{D}z$$

$$\therefore \qquad (x+1)\ \mathrm{D}z - nz = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!} = n_{\frac{1}{2}} \frac{x^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}}.$$

$$\therefore \qquad z = \frac{2n_{\frac{1}{2}}(z+x)^n}{\pi^{\frac{1}{2}}} \int \frac{dx^{\frac{1}{2}}}{(1+x)^{n+1}}.$$

In other words, we have the following generalised binomial series

$$1 + n_{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + n_1 \frac{x^1}{(1)!} + n_{\frac{3}{2}} \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + n_2 \frac{x^2}{(2)!} + \cdots$$

$$= (1+x)^n \left\{ 1 + \frac{2\, n_{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int \frac{dx^{\frac{1}{2}}}{(1+x)^{n+1}} \right\}.$$

9. *The generalised Exponential Series.*

Let $$z = \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \cdots$$

Proceeding as in § 8, we easily get

$$\mathrm{D}(z) - z = x^{\frac{1}{2}} / \pi^{\frac{1}{2}}.$$

$$\therefore \qquad z = \frac{2e^x}{\pi^{\frac{1}{2}}} \int e^{-x} dx^{\frac{1}{2}},$$

so that the series

$$1 + \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{x^1}{1!} + \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \cdots = e^x \left\{ 1 + \frac{2}{\pi^{\frac{1}{2}}} \int e^{-x}\, dx^{\frac{1}{2}} \right\}.$$

10. *The generalised Taylor Series.*

Here we put $$z = f^{\frac{1}{2}}(h) \frac{x^{\frac{1}{2}}}{(\frac{1}{2})!} + f^{\frac{3}{2}}(h) \frac{x^{\frac{3}{2}}}{(\frac{3}{2})!} + \cdots$$

and obtain $$\mathrm{D}_x z - \mathrm{D}_h z = \frac{x^{-\frac{1}{2}}}{\pi^{\frac{1}{2}}} \mathrm{D}_h^{\frac{1}{2}} f(h)$$

$$\therefore \qquad z = \frac{1}{\pi^{\frac{1}{2}}} \int^x d\xi . \xi^{\frac{1}{2}} \mathrm{D}_\xi^{\frac{1}{2}} f(x+h-\xi),$$

so that the generalised Taylor Series can be written

$$f(x+h) + \frac{1}{\pi^{\frac{1}{2}}} \int^x d\xi [\xi^{\frac{1}{2}} \mathrm{D}_h^{\frac{1}{2}} f(x+h-\xi)]$$

11. Consider the series,

$$z=\frac{x^a}{a\,!}+\frac{x^{a+1}}{(a+1)\,!}+\dots$$

where a may be any quantity except a negative integer.

We have
$$\mathrm{D}z=\frac{x^{a-1}}{(a-1)\,!}+\frac{x^a}{a\,!}+\dots$$
$$=\frac{x^{a-1}}{(a-1)\,!}+z.$$
$$\therefore\quad (\mathrm{D}-1)z=\frac{x^{a-1}}{(a-1)\,!}.$$
$$\therefore\quad z=\frac{e^x}{(a-1)\,!}\int x^{a-1}e^{-x}dx$$
$$=\frac{e^x}{a\,!}\int e^{-x}d(x^a).$$

The series z is obviously always convergent. Further, if we remember that

$$x^{1-a}z\cdot(a-1)\,!=\mathrm{F}(1,\beta,a,x/\beta),\text{ where }\beta\longrightarrow\infty$$

z can be represented by Barnes' integral as follows :

$$z=\frac{x^{a-1}}{2\pi i}\int_{-i\infty}^{\infty i}\frac{y\,!\,(\beta+y-1)!\,(-1-y)!}{(a+y-1)\,!\,(\beta-1)\,!}\left(-\frac{x}{\beta}\right)^y dy$$

where $\beta=\infty$;

12. *Properties of* $\left(x\,\frac{d}{dx}\right)$:

Let $\mathrm{D}_{\frac{1}{2}}$ denote $\left(x\frac{d}{dx}\right)^{\frac{1}{2}}$, then

$$\mathrm{D}_{\frac{1}{2}}(x^n)=\left(x\frac{d}{dx}\right)^{\frac{1}{2}}x^n=x^{\frac{1}{2}}n_{\frac{1}{2}}\,x^{n-\frac{1}{2}}$$
$$=n_{\frac{1}{2}}x^n.$$

Hence, we have

(i) $\mathrm{D}^r{}_{\frac{1}{2}}(x^n)=(n_{\frac{1}{2}})^r x^n$;

(ii) $\phi\left(\mathrm{D}_{\frac{1}{2}}\right)x^n=\phi\left(n_{\frac{1}{2}}\right)x^n$, from (i),

where ϕ is a polynomial.

13. Let λ_m denote the operation

$$\left[x^{\frac{1}{2}-m}\mathrm{D}_{\frac{1}{2}}^{1}x^m\right];$$

then $$\lambda_m(x^r) = x^{\frac{1}{2}-m} D^{\frac{1}{2}} x^{m+r}$$
$$= x^{\frac{1}{2}-m} (m+r)_{\frac{1}{2}} x^{m+r-\frac{1}{2}}$$
$$= (m+r)_{\frac{1}{2}} x^r.$$

Hence any number of operations λ_m, λ_n etc, are commutative.
[*Cf.* Edwards: *Diffl. Calc*, p. 486, Ex. 2.]

14. We can prove that when n is an integer,
$$D^n x^{n+\frac{1}{2}} D^{n+1} \div (x^{\frac{1}{2}} + \frac{1}{2^{2n+1}} \phi^{2n+1}(x^{\frac{1}{2}}).$$ [*Ibid*, Ex. 3]

For, the left hand side is eqnal to

$$D^n x^{n+\frac{1}{2}} D^{n+1} \Sigma(a_r x^{\frac{r}{2}}$$
$$= D^n x^{n+\frac{1}{2}} \Sigma \left\{ \left(\frac{r}{2}\right)_{n+1} a_r x^{(r/2-n-1)} \right\}$$
$$= D^n \Sigma \left(\frac{r}{2}\right)_{n+1} a_r x^{\frac{1}{2}(r-1)}$$
$$= \left(\frac{r}{2}\right)_{n+1} \left(\frac{r-1}{2}\right)_n a_r x^{\frac{r-1}{2}-n}$$

Also $$\frac{1}{2^{2n+1}} \phi^{2n+1}\left(x^{\frac{1}{2}}\right) :: \frac{1}{2_{2n+1}} \Sigma(r)_{2n+1} a_r x^{\frac{r-1}{2}-n};$$

and $$\left(\frac{r}{2}\right)_{n+1} \left(\frac{r-1}{2}\right)_n = \left(\frac{r}{2}\right)\left(\frac{r}{2}-1\right)\dots\left(\frac{r}{2}-n\right) \times$$
$$\left(\frac{r-1}{2}\right)\left(\frac{r-1}{2}-1\right)\dots\left(\frac{r-1}{2}-n+1\right)$$
$$= \frac{r(r-2)\dots(r-2n) \times (r-1)(r-3)\dots(r-2n+1)}{2^{2n+1}},$$
$$= \frac{1}{2^{2n+1}} \cdot (r)_{2n+1}.$$

Hence we obtain the result stated.

More generally, we can verify that

$$\frac{\left(\frac{r}{2}\right)!}{\left(\frac{r}{2}-n-1\right)!} \cdot \frac{\left(\frac{r-1}{2}\right)!}{\left(\frac{r-1}{2}-n\right)!} = \frac{1}{2^{2n+1}} \cdot \frac{r!}{(r-2n-1)!},$$

by the duplication formula. [Whittaker: p. 234, 12.15 Cor.]

The result may also be written in the form
$$2^r \left(\frac{r}{2}\right)! \left(\frac{r-1}{2}\right)! = \pi^{\frac{1}{2}} (r)!.$$

Hence, as before, we prove that the result quoted above is universally true, whatever n may be.

2° Partial Differentiation.

15. We shall, in this section, discuss the properties of partial differential symbols $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ &c, as applied to functions of more than one independant variable.

The following properties are immediate :

$$\text{(i)} \quad \frac{d^{m+n}}{dx^m dy^n} = \frac{d^{m+n}}{dy^n dx^m},$$

$$\text{(ii)} \quad \frac{d^{m+n+p}}{dx^m dy^n dz^p} = \frac{d^{m+n+p}}{dy^n dx^m dz^p} = \&c. ;$$

where m, n, p may be any quantities whatever.

16. *Euler's Theorems on Homogeneous Functions.*

These may be written (with our notation)

$$\text{(1)} \quad \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dots \right) u = n_1 u ;$$

$$\text{(2)} \quad \left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x}\partial y + y^2\frac{\partial^2}{\partial y^2} \dots \right) u = n_2 u,$$

and so on.

Thus, the generalized theorem would be

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} + \dots \right)^r u = n, u$$

whatever r may be, the left hand member being expanded as an ordinary multinomial.

These results may be further generalized as follows :—

(i) Let $u = x^m y^n$; then $D_1{}^a D^{\ b} u = m_a n_b u$, for all values of a if D_1 operates on x^m and D_2 on y^n.

Hence $$\phi\,(D_1, D_2)\, u = \phi(m, n)\, u,$$

where we are to understand that $(m^a n^b)$ is to be replaced by $(m_a n_b)$ in the expanded form of $\phi\ (m, n)$.

Again, if $$u = x^m y^n z^p,$$

we have $$\phi\,(D_1, D_2, D_3)\, u = \phi\,(m, n, p).\ u ;$$

and so on.

(ii) If u = a homogeneous expression of the rth degree, Euler's theorems will take the form

$$(D_1 + D_2 + \dots)\, u = (m + n + \dots) u = r\, u,$$

$$(D_1 + D_2 +)^{\ 2} u = r_2 u.$$

For fractional indices

$$(D_1 + D_2 + \dots)^{\frac{1}{2}}\, u = r_{\frac{1}{2}}.\ u = \frac{r\,!}{(r - \frac{1}{2})\,!}\, u,$$

and so on.

"On a system of three similar triangles, connected with a given triangle."

By S. Narayanan, Madura College.

1. Let A,B,C,D be four given points in a plane. It is known that the pedal triangles of each taken with respect to the triangle formed by the other three, are all directly similar, the similarity arising from the fact that the sides of each of the triangles are as the rectangles BC.AD, CA.BD, AB.CD. But it is perhaps not known that these four pedal triangles have the same double point. It will be shewn, in this paper, that they have a common double point which is the centre of the rectangular hyperbola ABCD.

From the property of the rectangular hyperbola, it is known that the pedal circles of each of the points A,B,C,D taken with respect to the triangle formed by the other three all pass through a common point. In this paper, quite apart from the geometry of conics, it is shown that these circles meet in a point and that point is the double point in question. These follow from a consideration of three similar triangles, formed in a particular manner, as explained below.

2. Let ABC be a given triangle and A_1 the foot of the perpendicular from A on BC. Drawing any line through A let BB_1, CC_1 be perpendiculars on it from B and C. Then it is readily seen from the figure that the triangle $A_1B_1C_1$ is inversely similar to ABC. Thus for every line drawn through A, there exists a triangle $A_1B_1C_1$, inversely similar to ABC. And a like construction gives, for every line through B as well as for every line through C, triangles inversely similar to ABC. We proceed to consider the triad of similar triangles arising when the lines through A, B, C *all pass through a given point* D.

3. Let the feet of the perpendiculars from B and C on A D be B_1 and C_1; those from C and A on BD be C_2 and A_2; and those from A and B on CD be A_3 and B_3. And let us denote the feet of the perpendiculars from A, B, C on B C, C A, AB respectively by A_1, B_2 and C_3. Then the triangles in question, corresponding to AD, BD' C D are $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$. These are directly similar to one another, each being inversely similar to ABC.

It may be noted at this stage, that $A_1A_2A_3$ is the pedal triangle of A with respect to the triangle BCD, $B_1B_2B_3$ is the pedal triangle of B, with respect to CDA and $C_1C_2C_3$ is that of C with respect to DAB.

4. **Theorem.** *In the similar triangles* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, *each set of corresponding sides is concurrent.*

To prove that B_1C_1, B_2C_2, B_3C_3 are concurrent.

Let B_2C_2, B_3C_3 meet in H_1. Let D′ be the isogonal conjugate of D in A B C. Now, B_2 and C_3 are the feet of the perpendiculars from B and C on the opposite sides in A B C. Hence the triangle A B_2C_3 is inversely similar to A B C. Also H_1 in A B_2 C_3 corresponds to D′ in A B C. For, the angle A C_3H_1 = B C_3B_3 = B C B_3 (for B, B_3, C_3, C are concyclic) = the angle B C D = A C D′. Similarly A B_2 H_1 = A B_2 C_2 = CBC_2 (for C, B_2, C_2, B are concyclic) = C B D = A B D′. Hence H_1 in A B_2C_3 corresponds to D′ in A B C as stated. Therefore the angle H_1AC_3 = D′ A C = B A D, which shews that H_1 lies on A D or B_1C_1. Thus B_1C_1, B_2C_2, B_3C_3 are concurrent at H_1.

In the same manner, the other sets of corresponding sides C_1A_1, C_2A_2, C_3A_3 concur at at H_2 and A_1B_1, A_2B_2, A_3B_3 concur at H_3.

5. Let us now specially consider the fact that, in the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, the corresponding sides intersecting *at the vertices* A_1, A_2, A_3 are concurrent, *viz.*, at H_2 and H_3. It follows that A_1, A_2, A_3, H_2, H_3 are all concyclic; for, as is seen from the figure, H_2H_3 subtends equal angles at A_2 and A_3 on one side and a supplementary angle at A_1 on the other side. *Hence we have the theorem that the pedal circle* $A_1A_2A_3$ *of* A *with respect to the triangle* BCD, *pass through the points* H_2 *and* H_3.

In like manner, the pedal circle $B_1B_2B_3$ of B with respect to the triangle CDA will pass through H_3 and H_1; and the pedal circle $C_1C_2C_3$ of C with respect to DAB will pass through H_1 and H_2.

Thus we see that H_1, H_2, H_3 constitute one set of the points of intersections of the three pedal circles taken two by two. We can *now* shew that the other points of intersections of these pedal circles taken in pairs are all one and the same point *w*; and that *w* is the common double point of tne similar triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

6. Let us take *w* to be the double point of the two triangles $A_2B_2C_2$ and $A_3B_3C_3$. Then wA_2 and wA_3 are corresponding lines of these triangles. They therefore divide the angles $H_2A_2H_3$ and $H_2A_3H_3$ similarly. Therefore their intersection *w* lies on the pedal circle of A with respect to the triangle BCD; and therefore *w* A_1 divides the angle $B_1A_1C_1$ similarly.

In like manner *w* B_2 and *w* B_3 being corresponding lines divide the angles B_2 and B_3 similarly and therefore *w* lies on the circle $H_3H_1B_1B_2B_3$;

that is w lies on the pedal circle of B with respect to the triangle CDA; and therefore w B_1 divides the angle B_1 similarly.

In like manner, again, w lies on the pedal circle $C_1C_2C_3$ H_1H_2 of C with respect to the triangle DAB; and w C_1 divides the angle C similarly.

We have thus shewn that the pedal circles all cointersect in w and that w holds, in respect of the triangle $A_1B_1C_1$ the same position that it holds in respect of the triangles $A_2B_2C_2$ and $A_3B_3C_3$; that is, *w is the common double point of the triangles* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

7. From the fact that the similar triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ have a common double point w, it follows that the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ which are formed by taking corresponding points of these similar triangles are themselves directly similar and have w for their common double point. And we proved that the circumcircles of these triangles all pass through w. These latter triangles are, as we already observed, the pedal triangles of A with respect to BCD, of B with respect to CDA, and of C with respect to DAB.

8. Taking now all the four pedal triangles of A, B, C, D, *viz*, of each taken with respect to the triangle formed by the other three, it follows that every three of them have a common double point and therefore all the four have a common double point w. And the circumcircles of all these pedal triangles pass through w. Hence w is the centre of the rectangular hyperbola ABCD.

9. We next proceed to shew that w lies on the nine-points circle of ABC; whence, it would follow that it lies also on the nine-points circles of each of the triangles BCD, CDA, DAB. To prove this, we revert to a consideration of the triangle $A_1B_1C_1$, a study of which, as the line through A varies, is interesting. For our present purpose, we just require one property of the triangle.

10. **Theorem**. *As the line through A varies, the locus of O_1, the circumcentre of A_1 B_1 C_1 is the nine-points circle of ABC.*

For, as the line through A varies, A_1 remains fixed and B_1 moves on circle AA_1B. the The triangle $A_1B_1C_1$ which may be considered to be described on A_1B_1 is of a known species. Hence any assigned point connected with it moves on a circle. In particular, the circumcentre of $A_1B_1C_1$ moves on a circle.

Now when the line through A coincides with AA_1, the triangle A_1 B_1 C_1 is evanescent and its circumcentre therefore coincides with A_1, a point on the nine-points circle of ABC; when the line through A

takes the position AB, the variable triangle $A_1 B_1 C_1$ assumes the position $A_1 B C_3$ and the circumcentre in this case is the middle point of B H (where H is the orthocentre of ABC) and hence in this case also the circumcentre of $A_1 B_1 C_1$ lies on the nine-points circle of ABC. Lastly, when the line through A coincides with AC, the variable triangle $A_1 B_1 C_1$ becomes $A_1 B_2 C$ whose circumcentre is similarly the middle pointof C H and therefore lies on the nine points circle of A B C. *Hence the locus of the circumcentre of the triangle* $A_1B_1C_1$ *is the nine-points circle of* A B C.

Let O_2, O_3 be the circumcentres of $A_2B_2C_2$ and $A_3B_3C_3$. It follows that these also lie on the nine-points circle of A B C.

Theorem. *Any set of corresponding circum-diameters of* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ *are concurrent, the point of concurrence lying always on the nine-points circle of* A B C.

For, the angle $B A_2 A_1 = B A A_1 = \frac{1}{2}\pi - B = C_2A_2O_2$. Therefore $A_1A_2O_2$ are collinear. Similarly $A_1A_3O_3$ are collinear. Hence O_1A_1, O_2A_2, O_3A_3 concur at A_1.

Similarly O_1B_1, O_2B_2, O_3B_3 concur at B_2 and O_1C_1 O_2C_2, O_3C_3 concur at C_3.

Now l_1, l_2, l_3 being any three corresponding diameters, the angles which A_1O_1, A_2O_2, A_3O_3 make with l_1, l_2, l_3 are equal. Hence l_1, l_2, l_3 concur at a point ϕ on the nine-points circle of A B C.

But w having been shewn to be the common double point of the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, the lines wO_1 wO_2, wO_3 are corresponding diameters. Therefore their intersection w lies on the nine-points circle of A B C.

In the same manner, w lies on the ninepoints circles of each of the other triangles BCD, CDA, DAB. Thus, all the nine-points circles cointersect in w.

12. Finally let m_1, m_2, m_3, denote *any* three lines drawn arbitrarily through A, B, C forming a triangle $D_1D_2D_3$; constructing the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ as before, let w_1 w_2 w_3 denote the double points of these three similar triangles taken two by two. Then it follows that they lie on the nine-points circle of ABC, being the points corresponding to D_1 D_2 D_3, in the same manner as w corresponding to D in the foregoing investigation. *Thus whatever the lines* m_1, m_2, m_3 *may be, the circle of similitude of these triangles is always the nine-points circle of ABC.*

It can further be shewn that the *invariable points* of these triangles are their *circumcentres* O_1, O_2, O_3.

The Theory of Linear Systems of Points.

R. VYTHYNATHASWAMY.

[The present theory is not exactly new. In its root principles it will be found identical with the Invariant theory and the theory of Polynomial forms. But the view-points are different. While in the theory of invariants and of binary forms, it is the forms themselves and the relations between forms which are the subjects of study, the centre of interest, here, is the system of forms (or what is the same thing, of groups of points) all possessing a common property. That this displacement of the point of view does bring about a difference of methods and conceptions will be seen in the course of the paper.

With reference to the harmonic property, the chapter on 'Apolarity' in Grace and Young's *Algebra of Invariants* might be consulted with advantage. The difference in nomenclature is explained by the fact that this paper was written before I had any knowledge of the the Apolar Theory.

The principle of the geometrical representation is the same as in the representation of point-pairs in a line by points in a plane explained in my Note on 'Involution and 1, 1 Correspondence'.]

1. *Defintion* : The groups of points in a line whose coordinates satisfy k lineo-linear equations of the type

$$F \equiv a_0 x_1 x_2 \ldots x_n + a_1 \Sigma(x_1 x_2 \ldots x_{n-1}) + a_2 \Sigma(x_1 x_2 \ldots x_{n-2}) \ldots = 0$$

will be said to form a linear system of order n and class $n-k+1$.

A system of order n and class n (*i.e.* a system defined by a single equation) will be called a *complete* system. Linear systems of order n and class other than n, will be termed *incomplete*. The involution of point-pairs, for example, is a complete linear system of order 2.

We may give another definition of a linear system which is descriptive in character and which will be found more useful in practice. If we have a system of groups of n points which is such that there is an absolute symmetry among the points composing any group, and that if any $r-1$ points of a group are given, the rest of the group is *in general* uniquely determined, then the system is a linear system of order n and class r, In particular, if, when all but one of the points of any group are known, the group is determined, the system is a comple system. Thus groups of points on a nodal cubic which are its intersections with a line, a conic and a cubic form complete systems of orders 3, 6 and 9 respectively.

2. *Geometrical Interpretation.*

Represent every group of n points by a point in space of n dimensions. If the group be $(x_1\ x_2 \ldots x_n)$, the co-efficients of the equation whose roots are $x_1, x_2 \ldots x_n$ may be taken to be the homogenous co-ordinates of the representative point. It may be noted that the representative point depends only on the group and not on the order of points group.

In this representation a complete system would correspond to a *plane*. Here and in what follows a plane in n dimensions means a flat space of $n-1$ dimensions. An incomplete system of class r would be represented by a flat space of $r-1$ dimensions. An immediate inference from this geometrical representation is that a unique system of class r can be found to contain r given n—ads. In particular, a complete system of order n is determined by n groups.

3. There are n n—ple points belonging to the complete system

$$\mathrm{F} \equiv \sum a_r \Sigma x_1 x_2 \ldots x_{n-r} = 0$$

of order n, these points being evidently the roots of the equation

$$\phi \equiv a_0 x^n + n\, a_1 x^{n-1} \ldots\ldots\ldots + a_n = 0.$$

These points will be termed the foci of the system. It is obvious that an incomplete system will not in general possess any foci.

A complete system is determined by its foci.

(1) If all the foci of a complete system coalesce at K, the system consists of groups composed of K and any other $n-1$ points.

For, if ϕ is an exact nth power $a_0 (k-k)^n$, then

$$\mathrm{F} \equiv a_0 (x_1 - k)(x_2 - k) \ldots\ldots (x_n - k).$$

(2) If the complete system whose foci are $x_1\ x_2 \ldots x_n$ contains the group $y_1\ y_2 \ldots y_n$, then the complete system whose foci are $y_1\ y_2 \ldots y_n$, contains the group $x_1\ x_2 \ldots x_n$.

Let $x_1\ x_2 \ldots x_n$, $y_1\ y_2 \ldots y_n$ be respectively roots of the equations

$$a_0 x^n + n\mathrm{C}_1\, a_1 x^{n-1} + n\mathrm{C}_2\, a_2 x^{n-2} \ldots + a_n = 0, \quad \ldots \quad \ldots \quad (1)$$

$$b_0 y^n + n\mathrm{C}_1\, b_1 y^{n-1} \ldots\ldots\ldots\ldots + b_n = 0. \quad \ldots \quad \ldots \quad (2)$$

The system whose foci are the roots of (1) contains the roots of (2). Hence

$$(-)^n (a_0 b_n - n\mathrm{C}_1\, a_1 b_{n-1} + n\mathrm{C}_2\, a_2 b_{n-2} \ldots) = 0.$$

The symmetry of the a's and b's in this relation proves the theorem.

The point-groups $x_1 x_2 \ldots x_n$, $y_1 y_2 \ldots y_n$ of this theorem may be said to be *harmonic*. Thus a complete system would consist of all groups which are harmonic to the focal group.

(3) A complete system of odd order contains its focal group.

Proceeding as above, we find the condition that the complete system whose foci are $x_1 x_2 \ldots x_n$, should contain the group $x_2 x_2 \ldots x_n$ to be

$$(-)^n (a_o a_n - {}^nC_1 a_1 a_{n-1} + \ldots\ldots) = 0$$

This equation is *identically* true if n is odd. Hence the theorem.

This theorem is tantamount to saying that every odd group is self-harmonic.

(4) The harmonic relation is invariant for linear transformations.

For, a complete system being defined by a lineo-linear equation, is transformed into another such system and the focal group is transformed into the focal group.

(5) If r points of a group of the nth order coalesce at K, the group is harmonic to a group consisting of $n-r+1$ points at K and *any other* $r-1$ points.

Make a homographic transformation reducing K to zero. If now the points of the two groups are roots of the equations

$$a_o x^n + {}^nC_1 a_1 x^{n-1} \ldots\ldots + {}^nC_{n-r} a_{n-r} x^r = 0,$$
$$b_o x^n + {}^nC_1 a_1 x^{n-1} \ldots\ldots + {}^nC_{r-1} a_{r-1} x^{n-r+1} = 0,$$

using the condition of (2), the two groups are seen to be harmonic for any values of a's and b's.

(6) An even group is self-harmonic if more than half of its points coalesce.

This follows directly from (5).

4. *The Indeterminate Systems and the Extension of the Harmonic Relation.*

Any two groups of points forming together a group of a given complete system may be said to be conjugate. All groups conjugate to a given group of r points will form a complete system of order $n-r$, which may be termed the polar system of the given group. The polar system of the same group in this latter system is its second polar system and so on.

It is obvious on inspection that the first, second,.....systems of the point at infinity *w. r. t.* the complete system

$$F \equiv a_o x_1 x_2 \ldots x_n + a_1 \Sigma x_1 x_2 \ldots x_{n-1} + \ldots\ldots + a_n = 0,$$

are given by $D(F)=0,\ D^2(F)=0,\ldots\ldots D^r(F)=0,$

where D denotes differentiation *w. r. t.* one of the x's.

(1) The foci of the systems $D^k(F)$ are the successive harmonic centres of the foci of F. (*vide*: Mr. Naraniengar's *Note on the Harmonic Centres*, J.I.M.S., Feb. 1913.)

For let $$\phi = a_o x^n + n C_1 a_1 x^{n-1} + \ldots\ldots\ldots + a_n.$$

It is evident on inspection that the foci of D(F) are the roots of

$$\frac{d\phi}{dx} = 0, \text{ or } \Sigma \frac{1}{x-\rho_r} = 0,$$

where $\rho_1\ \rho_2$.....are the roots of $\phi = 0$.

Thus the foci of D(F) are the harmonic centres of the foci of F. Repeating the proof, the foci of $D^2(F)$ are the harmonic centres of the foci of D(F) and so on.

(2) If F_1, F_2.....represent the successive polar systems of the origin, then the polar system of the group $t_1 t_2 \ldots t_r$ is given by

$$F_r + D(F_{r-1})\Sigma(t_1) + D^2 F_{r-2}\Sigma(t_1 t_2) \ldots\ldots + D^r(F).\ t_1 t_2 \ldots t_r = 0.$$

For $$F \equiv F_1 + x_1 D(F).$$
$$\equiv F_2 + x_2 D F_1 + x_1 (D F_1 + x_2 D^2 F)$$
$$\equiv F_2 + D F_1 (x_1 + x_2) + x_1 x_2 D^2 F;$$

and so on.

(3) The polar system of all r-ads of points *w. r. t.* a complete system of order n, have in common a system I_r of order $n-r$ and class $n-2r$, this system evidently consisting of the point-groups common to the complete systems

$$F_r,\ D(F_{r-1}),\ D^2 F_{r-2} \ldots\ldots D^r(F).$$

The conjugate group of any group of the system I_r is indeterminate and may be supposed to be any r points whatever. For this reason the systems I_r will be termed the *indeterminate systems* of F.

The system I_r will reduce to a single group if $n-2r=1$. Thus the number of the systems I_r is the greatest integer in $\frac{1}{2}(n-1)$ and if n is odd there will be a single group of $\frac{1}{2}(n+1)$ points whose conjugate group is indeterminate. Since the groups of a complete system are harmonic to the focal group, it follows that a group of the system I_r together with *any* r points is harmonic to the focal group of F.

We might agree to express this by saying that the groups of the system I_r are harmonic to the focal group of F. This is the natural *extension* of the harmonic relation to groups of unequal order.

(5) *Indeterminate Groups of Incomplete Systems.*

An incomplete system of order n and class δ—or, as it will be written, of type $[n, \delta]$—may be regarded as the intersection of $n-\delta+1$ complete systems of order n. An indeterminate group of order $n-r$ of $[n, \delta]$ is therefore a common group of the $n-\delta+1$ systems I_r of the complete systems. Since I_r is of class $n-2r$, it is evident that the $n-\delta+1$ systems I_r have in common a system I_r' of order $n-r$ and class $\{n-r-(n-\delta+1)(r+1)+1\}$ or $[\delta-r(n-\delta+2)]$ The number of the systems I_r' is therefore the greatest integer in $\frac{\delta-1}{n-\delta+2}$. If $\frac{\delta-1}{n-\delta+2}$ is an exact integer μ, we will have a group of $n-\mu$ points whose conjugate is absolutely interminate.

The following is a table showing the indeterminate groups of systems of orders up to 10. A single indeterminate group of order, say 4, is written $4^{(1)}$, while the existence of a system of indeterminate groups of, say order 4 and class 2, is denoted by $4^{(2)}$.

Incomplete systems of order less than 5 have no indeterminate groups.

Table of Indeterminate Systems.

δ	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
4	$4^{(1)}$					
5	$4^{(3)}, 3^{(1)}$	$5^{(2)}$	$6^{(1)}$			
6		$5^{(4)}, 4^{(2)}$	$6^{(2)}$	$7^{(2)}$	$8^{(1)}$	
7			$6^{(5)}, 5^{(3)}$ $4^{(1)}$	$6^{(1)}, 7^{(4)}$	$8^{(3)}$	$9^{(2)}$
8				$7^{(6)}, 6^{(4)}$ $5^{(2)}$	$7^{(2)}, 8^{(5)}$	$9^{(4)}$
9					$8^{(7)}, 7^{(5)}$ $6^{(3)}, 5^{(1)}$	$8^{(3)}, 9^{(6)}$

6. We proceed to the discussion of a very important class of incomplete systems called *focal systems*.

(1) *A system of class r cannot have more than r foci.* If possible, let K be a system of class r and order n ($> r$) containing more than r foci. We can find a unique complete system containing K and any other $n-r$ given n-ads (Compare the geometric interpretation of § 2) not belonging to K. In particular, we can find a complete system containing K and having $n-r$ points for foci. If K has more than r foci, this complete system will have more than n foci, which is impossible.

Def. A system of class r containing r foci is a *focal system.*

A system of class r is completely determined by r groups and therefore by its r foci.

Note that every complete system is a focal system.

(2) *The focal group of a focal system is harmonic to every group of the system.*

For, we can find a complete system containing K and having any other, $n-r$ points as foci. Therefore every group of K is harmonic to the group formed by any $(n-r)$ points and the focal group of K. Thus the focal group of K is an indeterminate group of every complete system whose focal group is a group of K. From our extended definition of the harmonic relation, it follows that the focal group is harmonic to every group of the eystem.

Cor. A unique focal system of class r can be found to contain a given group of $2r-1$ points. For, by § 3 (3) there is a uniqne indeterminate group of r points belonging to the complete system whose foci are the $2r-1$ points.

It will be evident that either no focal system or an infinite number of focal systems of class r can be found to contain a given group of n points according as $n>$ or $<2r-1$.

7. *Focal systems of class 2.*

Theorem. *The focal system of class 2 and order n whose foci are f_1 f_2 consists of all cyclic groups associated with the homographic correspondence of period n whose fixed points are f_1 f_2.*

Since a focal system is completely determined by its foci, it suffices to prove the converse, *viz.* that the cyclic groups of such a homographic correspondence do form a system of class 2 with foci at f_1, f_2.

Let f_1, f_2 be $0, \infty$. Then the homographic correspondence of period n with these as fixed points will reduce to $x^1 = \omega x$ where $\omega^n = 1$. Thus the cyclic groups are roots of the equation $x^n = \lambda$ where λ is a parameter. Hence such groups form a linear system of class 2. Evidently here are values of λ (*viz* $0, \infty$), for which all the roots of the above

equation become equal. Hence this linear system is a focal system and has foci at $0, \infty$. This proves the theorem.

It is not difficult to see that this theorem is not a special one but only a particular case of a more general theorem which connects every group of a focal system with the focal group by means of a periodic *rational*, or more probably, *algebraic transformation*. But it is impossible to get even a glimpse of this general theorem in our present state of knowledge of Rational and Algebraic Functions.

8. *Expression of a n-tic as a linear combination of nth powers* (*Vide*:—Grace and Young : *Algebra of Invariants*, Chapter on 'Apolarity'.)

Since a system of class r is represented by a *flat* space of $r-1$ dimensions, it follows that any group of the system (*i.e.* the polynomial whose roots are the points of the group) can be expressed as a linear combination of r fixed groups of the system. In particular, if a focal system with foci at $\alpha, \beta\ \gamma \ldots$ can be found to contain a given group of order n, then the polynomial whose roots are the points of the group can be expressed as a linear combination $(x-\alpha)^n$, $(x-\beta)^n$, $(x-\gamma)^n$ etc. Now by § 6. (2) Cor., if n is of the form $2r-1$, there is a unique focal system of class r which contains the group, and if n is of the form $2r$, there is an one-fold infinity of focal systems of class $(r+1)$ containing the group.

Hence a $(2r-1)$—tic can be expressed uniquely as the sum of r terms of the $(2r-1)$ th degree and a $2r$—tic can be expressed in a singly infinite number of ways as the sum of $(r+1)$ powers of the $2r$th degree.

9. *Geometrical Interpretation.*

We represented every group of n points in a line by a point in n-dimensional space and we found that a complete system was represented by a plane. We may now enquire what points in our n—space will correspond to a special type of n—groups, *viz.*, those groups in which all the points coalesce. The locus of such points in n-space will, in the first place, be a curve and this curve should cut every plane in n points, since there are n coalesced groups or foci in every complete system.

Thus the locus of points corresponding to coalesced n—groups is a curve of nth degree in n dimensions.

This curve will be called the twisted n—tic and is the natural extension of the conic and the twisted cubic.

If $X_1 X_2 \ldots X_n$ are points on the twisted n—tic corresponding to the points $x_1 x_2 \ldots x_n$ on the line, the complete system whose foci are $x_1 x_2 \ldots x_n$ is represented by the plane $X_1 X_2 \ldots X_n$.

We have pointed out (§ 3 (5)) that if all the foci of a complete system coalesce, the system consists of groups composed of the point of coalition and any other $n-1$ points.

Hence the locus of all groups which contain a point x_1 is the osculating plane at X_1.

Thus the group $x_1 x_2 \ldots x_n$ is represented by the point of intersection of the osculating planes at $X_1 X_2 \ldots \ldots X_n$. (We see incidentally that the class of the twisted n—tic is n).

If P is any point in n—space, we may define the polar plane of P as the plane determined by the points of contact of the n osculating planes which can be drawn from P. Hence, from what has been stated above it follows that

(1) the complete system whose focal group corresponds to P, is represented by the polar plane of P ;

(2) the focal group of any complete system is represented by the polar point of the corresponding plane.

We have proved that a complete system of odd order contains its focal group and a complete system of even order contains its focal group, only if a certain relation of the second degree be satisfied. *Hence in spaces of odd dimensions, every plane contains its polar point (or the polar plane of any point P passes through P). In spaces of even dimensions the polar plane of P does not pass through P unless P lies on a certain quadric surface which will be called the special quadric associated with the twisted n—tic.* It is easily seen that the special quadric contains the twisted n—tic.

Again, it was proved that if more than half of the points of an even group coalesce the group is self-harmonic. *Hence in space of 2m dimensions osculating regions of m—1 dimensions are generating regions of the special quadric.*

Again, consider an even group $x_1 x_2 \ldots \ldots x_{2m}$. The polar plane of the corresponding point *w. r. t.* the *special quadric*, is a plane containing all self-harmonic groups harmonic to the given group. But $\{x_r x_r \ldots \ldots$ $2m$ times $\}$ is a self-harmonic group which is harmonic to $x_1 x_2 \ldots x_{2m}$. Hence the polar plane *w. r. t.* the special quadric contains the points $X_1 X_2 \ldots \ldots X_{2m}$ which are points on the n—tic corresponding to $x_1 x_2 \ldots x_{2m}$

In other words, *in space of even dimensions, polarisation w. r. t the twisted n-tic is identical with polarisation w. r. t. the special quadric.*

It must be remembered in connection with this result that the special quadric exists only in even dimensions. In space of odd dimensions, we may guess that polarisation *w. r. t.* the twisted n—tic would be identical with polarisation *w. r. t.* something which is similar in nature to the linear complex in three dimensions.

Lastly, the symmetry of the harmonic relation shews that: *if the polar plane of A passes through B, the polar plane of B passes through A*— a result valid for all dimensions.

Representation of groups of order less than n in n dimensions.

The group $x_1x_2...x_r$ $(r<n)$ can be represented in n-space only as the region of intersection of the osculating planes at $X_1X_2...X_r$. This is, in effect, equivalent to regarding the group as identical with the linear system of groups composed of $x_1x_2...x_r$ and any other $n-r$ points. A region of this type we shall term '*a region in r planes*'.

The indeterminate system I_r of a complete system K will correspond to the aggregate of regions in $(n-r)$ planes contained by the plane corresponding to K.

In particular, if n is odd, every plane in n dimensions contains a a unique region in $\frac{1}{2}(n+1)$ planes.

10. *Focal Systems.*

A focal system of the rth class is completely determined by its r foci. Such systems will therefore correspond to regions of $r-1$ dimensions which cut the twisted $n-$tic in r points.

If n is odd, then through any point P there can be drawn a unique region of $\frac{1}{2}(n-1)$ dimensions cutting the $n-$tic in $\frac{1}{2}(n+1)$ points [§ 6 (2) cor.]

As an example take a triad A B C and let p be the corresponding point in three-space. The triad A B C has a unique indeterminate pair which is the focal pair of the unique focal system containing ABC and which is also (§ 7.) the pair of fixed points of the homographic correspondence $\begin{vmatrix} A & B & C \\ B & C & A \end{vmatrix}$. This pair I have elsewhere termed the cyclic centres of ABC. (*Vide* : Note on Pascal Hexagon, J.I.M.S., June, 1918). This pair is represented in three-space as the points of intersection with the twisted cubic of the unique chord of the curve which can be drawn through P.

SHORT NOTES.

Addition Formulae for sn $(u_1+u_2+u_3)$ &c.

The symmetrical fractional expressions for the sn, cn, dn functions of $(u_1+u_2+u_3)$ given by me in the J. I. M. S., Vol. VIII p. 211, when reduced to their lowest terms, are identical with Professor Cayley's expressions quoted in Ex. 20, page 523 of Whittaker and Watson's *Modern Analysis* (Revised Edition). I give below the analysis required for this reduction, which supplies an elegant proof of Prof. Cayley's results and must interest those who, like myself, have not had the opportunity of seeing the original proof.

If s_r, c_r, d_r stand as usual for sn u_r, cn u_r, dn u_r; s_r', c_r' d_r' stand for

$$\frac{\partial}{\partial u_r}(s_r),\ \frac{\partial}{\partial u_r}(c_r),\ \frac{\partial}{\partial u_r}(d_r)$$

respectively and P stand for

$$\text{sn}\,(u_2-u_3)\ \text{sn}\,(u_3-u_1)\ \text{sn}\,(u_1-u_2),$$

we can prove by Liouville's theorem that

$$\text{sn}\,(u_1-u_4)\ \text{sn}\,(u_2-u_3)+\text{sn}\,(u_2-u_4)\ \text{sn}\,(u_3-u_1)+\text{sn}\,(u_3-u_4)\ \text{sn}\,(u_1-u_2)$$
$$\equiv -\text{P}.\ k^2\ \text{sn}\,(u_1-u_4)\ \text{sn}\,(u_2-u_4)\ \text{sn}\,(u_3-u_4). \quad \ldots \quad \text{(A)}$$

Now writing successively $u_4=0$, iK', K, $K+iK'$, we have

$$\Sigma\, s_1\ \text{sn}\,(u_2-u_3)=-\text{P}.\ k^2\ s_1\ s_2\ s_3, \quad \ldots \quad \ldots \quad \text{(A·1)}$$
$$\Sigma\, s_2\ s_3\ \text{sn}\,(u_2-u_3)=-\text{P} \quad \ldots \quad \ldots \quad \text{(A·2)}$$
$$\Sigma\, c_1\ d_2\ d_3\ \text{sn}\,(u_2-u_3)=-\text{P}\ k^2\ c_1\ c_2\ c_3 \quad \ldots \quad \ldots \quad \text{(A·3)}$$
$$\Sigma\, d_1\ c_2\ c_3\ \text{sn}\,(u_2-u_3)=-\text{P}.\ d_1\ d_2\ d_3 \quad \ldots \quad \ldots \quad \text{(A·4)}$$

the summation extending to the suffixes 1, 2, 3.

Again writing in (A.1), $u_1+u_2+u_3=3x$, $u_1-u_2=3y$, $u_1-u_3=3z$, and consequently writing $u_2-u_3=3(z-y)$, $u_1=x+y+z$, $u_2=x-2y+z$, $u_3=x+y-2z$, and differentiating with respect to x, we have

$$\Sigma c_1 d_1\ \text{sn}\,(u_2-u_3)=-\text{P}\ k^2\Sigma s_2 s_3 s_1'. \quad \ldots \quad \text{(A. 5.)}$$

The last result gives rise to three others when u_r+K, u_r+iK' or u_r+K+iK' is substituted for u_r. For instance, writing u_r+K+iK' for u_r we have

$$\Sigma s_1 c_2^2 c_3^2\ \text{sn}\,(u_2-u_3)=-\text{P}\ \Sigma s_1 s_2' s_3' \quad \ldots \quad \ldots \quad \text{(A. 6)}$$

2. Again since $s_1^n=s_1^{n-2}\Sigma s_1^2-s_1^{n-2}\,[1+s_2^2 s_3^2-c_2^2 c_3^2]$, we have

$$\Sigma s_1^3\ \text{sn}\,(u_2-u_3)=\Sigma s_1 c_2^2 c_3^2\ \text{sn}\,(u_2-u_3)$$
$$-(1-\Sigma s_1^2)\ \Sigma s_1\ \text{sn}\,(u_2-u_3)-s_1 s_2 s_3\ \Sigma s_2 s_3\ \text{sn}\,(u_2-u_3)$$
$$=-\text{P}\ [\Sigma s_1 s_2' s_3'-s_1 s_2 s_3\ (1+k^2-k^2\Sigma s_1^2)]$$

from (A. 6), (A. 1) and (A. 2) $\quad \ldots \quad \ldots \quad \ldots \quad \text{(B. 1)}$

Now writing u_1+iK' for u_1, we have

$$\Sigma s_2's_3 \text{ sn } (u_2-u_3)=-P[s_1s_2s_3\ \Sigma s_1s_2's_3'-s_1^2s_2^2s_3^2.\ (1+k^2)+\Sigma s_2^2s_3^2].$$

It may be noted here that $\Sigma s_1^n \text{ sn } (u_2-u_3)$ is divisible by P when n is odd, and that if P. $f(n)\equiv\Sigma s_1^n \text{sn } (u_2-u_3)$, we can deduce the following reduction formula for $f(n)$, *for odd values of* n *greater than* 3:

$$f(n)+(1-\Sigma s_1^2)\ f(n-2)+s_1^2s_2^2s_3^2\ f(n-4)$$
$$=-\{\Sigma s_1s_2's_3'+c_1^2c_2^2c_3^2\ [f(1)+f(3)+\ldots+f(n-4)]\}. \quad \text{(B. 3)}$$

3. If X, Y, Z, stand for the different numerators and W for the common denominator of the expressions for sn, cn, dn of $(u_1+u_2+u_3)$ as given in J. I. M. S. (Vol. VIII, p. 211,), we have

$$\begin{aligned}
X &= \Sigma s_1's_2s_3\ (s_2^2-s_3^2)=\Sigma s_1^2(s_2s_3'-s_3s_2') \\
&= \Sigma[s_1^2(1-k^2s_2^2s_3^2) \text{ sn } (u_2-u_3)] \\
&= \Sigma s_1^2 \text{ sn } (u_2-u_3)-k^2s_1^2s_2^2s_3^2\Sigma s_1 \text{ sn } (u_2-u_3) \\
&= -P[\Sigma s_1s_2's_3'-s_1s_2s_3(1+k^2-k^2\Sigma s_1^2+k^4s_1^2s_2^2s_3^2)] \\
&\qquad \text{from (A.1) and (B.1)} \quad \ldots \quad \text{(C.1)}
\end{aligned}$$

$$\begin{aligned}
Y &= -\Sigma c_1'c_2c_3(c_2^2-c_3^2)=-c_1^2(c_2c_3'-c_3c_2') \\
&= -\Sigma[c_1^2(d_2d_3-k^2s_2s_3c_2c_3) \text{ sn } (u_2-u_3)] \\
&= -\Sigma\left\{\left[\left(\frac{d_1^2}{k^2}-\frac{k'^2}{k^2}\right)c_1d_2d_3-k^2c_1c_2c_3s_2s_3(1-s_1^2)\right]\right. \\
&\qquad \left.\text{sn } (u_2-u_3)\right\} \\
&= -\frac{1}{k^2}d_1d_2d_3\Sigma c_1d_1 \text{ sn } (u_2-u_3)+\frac{k'^2}{k^2}\Sigma c_1d_2d_3 \text{ sn}_4^2(u_2-u_3) \\
&\quad +k^2c_1c_2c_3\Sigma s_2s_3 \text{ sn } (u_2-u_3) \\
&\quad -k^2s_1s_2s_3\ c_1c_2c_3\Sigma s_1 \text{ sn } (u_2-u_3) \\
&= -P[c_1c_2c_3(1-k^4s_1^2s_2^2s_3^2)-d_1d_2d_3\Sigma s_2s_3s_1' \\
&\qquad \text{from (A.5), (A.3), (A.2), (A.1)}] \ \ldots \quad \text{(C.2)}
\end{aligned}$$

$$\begin{aligned}
Z &= -\frac{1}{k^4}\Sigma d_2d_3d_1'\ (d_2^2-d_3^2) \\
&= -\frac{1}{k^2}\Sigma[d_1^2(c_2c_3-s_2s_3d_2d_3) \text{ sn } (u_2-u_3)] \\
&= -\frac{1}{k^2}\ \Sigma\left[d_1c_2c_3\ (k^2+k^2c_1^2) \text{ sn } (u_2-u_3)\right] \\
&\quad +\frac{1}{k^2}\ \Sigma\left[d_1d_2d_3\ s_2s_3\ (1-k^2s_1^2) \text{ sn } (u_2-u_3)\right] \\
&= -\frac{k^2}{k^2}\Sigma d_1c_2c_3 \text{ sn } (u_2-u_3)-c_1c_2c_3\ \Sigma c_1d_1 \text{ sn } (u_2-u_3) \\
&\quad +\frac{1}{k^2}d_1d_2d_3\Sigma s_2s_3 \text{ sn } (u_2-u_3)
\end{aligned}$$

$$-d_1 d_2 d_3 s_1 s_2 s_3 \Sigma s_1 \operatorname{sn}(u_2-u_3)$$

$$= -P\left[d_1 d_2 d_3 (1-k^2 s_1^2 s_2^2 s_3^2) - k^2 c_1 c_2 c_3 \Sigma s_1 s_2 s_3\right]$$

from (A. 4), (A. 5), (A. 2), (A. 1) ... (C.3)

$$W = -\Sigma [s_1 s_1^2 (s_2^2 - s_3^2)]$$
$$= \Sigma [s_2 s_3 (1-k^2 s_2^2 s_3^2) \operatorname{sn}(u_2-u_3)]$$
$$= \Sigma s_2 s_3 \operatorname{sn}(u_2-u_3) - k^2 \Sigma s_2^3 s_3^3 \operatorname{sn}(u_2-u_3)$$
$$= -P[1-k^2 \Sigma s_2^2 s_3^2 + (k^2+k^4) s_1^2 s_2^2 s_3^2 - k^2 s_1 s_2 s_3 \Sigma s_1 s_2' s_3']$$

from (A. 2) and (B. 2) ... (C.4).

From (C. 1), (C. 2), (C. 3), (C. 4), are derived Prof. Cayley's results.

F. H. V. GULASEKHARAM.

On Question 963.

In this question Prof. Wilkinson has given a very interesting property connected with quadrilaterals which are inscribed in one circle and circumscribed to another. The property is stated however in a manner which leads one to surmise that Prof. Wilkinson, at the moment, must have been in the fighting line of thought, leaving *expression* to take care of itself. Mr. F. H. V. Gulasekharam has, in the last number of the Journal, supplied the commentary required to understand the theorem, and also a proof *bringing out other noteworthy properties of the figure.* On reading it, I was tempted to see if the proof could not be simplified,—for a neat theorem is generally capable of a neat proof. In my attempt, I find that the property noticed by Prof. Wilkinson holds for polygons P, *of any number of sides,* which are inscribed in one given circle X and circumscribed to another Y. If Q denote the polygon whose successive vertices are the points of contact of the successive sides of P, then, the areas of P and Q remain in a constant ratio, as P varies under Poncelet's theorem.

In particular, the theorem is true in the case of *triangles*, a matter which I had not known previously, but which is readily verified.

2. To prove the general theorem: Let ABC...KL be the polygon P inscribed in the circle X; and let LA, AB, BC,...KL touch the circle Y in A′, B′, C′,...L′. Let I denote the centre of the circle Y. Then I A′, I B′,...are perpendiculars to the corresponding sides of P; and

therefore every adjoining two of them (as I A′, I B′) include an angle which is the supplement of the corresponding angle (A) of the polygon P. Now the polygon is the sum of n isosceles quadrilaterals like IA′ AB′. If $t_a, t_b, \ldots t_l$ denote the length of the tangents from A, B, ...L to circle Y and r denote its radius, then the area of the quadrilateral is evidently $r.t_a$. The area of the polygon P is therefore $r\,\Sigma t_a$. ... (1)

The area of the polygon Q is the sum of n isosceles triangles like I A′B′. Hence an expression for its area is $\frac{1}{2} r^2 \Sigma \sin A$, where A denotes the angle at A of the polygon P. Now sin A = chord LB/2 R, where R is the radius of the circle X. Hence the area of the polygon Q is $\frac{r^2}{4R}\,\Sigma$ (chord LB), where the summation extends to all chords joining two *alternate* vertices of P. ... (2)

By coaxal circle theory all such chords are tangents to a third circle, call it Y′, which is coaxal with X and Y ; and, it may be observed is the same for every polygon P. Denoting the tangents to this circle Y′ from the vertices A, B,...L by $t'_a, t'_b, \ldots t'_c$, we obviously have chord $LB = t'_c + t'_b$.

Hence the area of

$$Q = \frac{r^2}{4R} \sum (t'_c + t'_b) = \frac{r^2}{2R} \Sigma t'_a \qquad \ldots \qquad (3)$$

Again by coaxal circle theory, if t, t' be the tangents from any point on the circle X to the circles Y, Y′ the ratio t/t' is a constant, say k, which is dependent merely on the circles.

Hence
$$\frac{t_a}{t'_a} = \frac{t_b}{t'_b} = \ldots = \frac{t_l}{t'_l} = k,$$

which gives
$$\frac{\Sigma t_a}{\Sigma t'_a} = k, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

$$\therefore \quad \frac{\text{area of P}}{\text{area of Q}} = \frac{r.\,\Sigma t'_a}{r^2/2R.\,\Sigma t'_a} = \frac{2R}{r}.\,k, \text{ which is constant.}$$

(RAMNAD,)
13th May '19.

V. RAMASWAMI AIYAR.

Astronomical Notes.

1. *Change in the commencement of the Astronomical day* :—

It has been decided to substitute in nautical publications a day commencing at midnight, for the astronomical day which begins at noon. The necessary alterations will be made in the Nautical Almanac from the year 1926.

2. *Planet Notes* :—

Venus is a conspicuous object in the Western sky during these months. It will be at greatest elongation on July 5 and attain greatest brilliancy on August 7.

The positions of planets on July 1 will be as below :—

		R. A. h.	m.	Declination	Constellation
Mercury	...	8	6	22°.0 N	Cancer.
Venus	...	9	47	14.8 N	Leo
Mars	...	5	39	23.9 N	Gemini
Jupiter	...	7	39	21.8 N	Cancer
Saturn	...	9	52	14.5 N	Leo
Uranus	...	22	15	11.7 S	Pisces
Neptune	...	8	41	18.2 N	Cancer

It is interesting to note that all the planets except Uranus will occupy the same region of the sky between R.A. 5 h. and 10 h.

Venus will be in conjunction with Saturn on July 2.

Rev. T. E. R. Phillips has found that some remarkable changes have recently occurred on the surface of Jupiter. The South Tropical Disturbance and the Hollow in the South Equatorial Belt disappeared in April but the Red Spot remains quite distinct.

3. *Absolute Magnitudes of Stars.*

In the *Observatory* May 1919 there is a note giving a clear explanation of the term. The *absolute magnitude* is defined as the magnitude which the star would have if seen from a distance of 10 *parsecs*, that is, from a distance equivalent to a parallax of a tenth of a second in arc. The Sun's absolute magnitude on this scale is $5^m.1$, that is, when removed to a distance of 10 *parsecs* the sun will shine as a star of 5.1 magnitude.

The relation between absolute magnitude M and luminosity L of a star is given by $\log L = 0.4\,(5.1 - M)$,
where the luminosity of the sun is chosen as unit.

4. *Cepheid Variables* :—

These Variables are so named after δ Cephei the typical star of the class. They exhibit variation in brightness which is regular and continuous, the rise to maximum being much more rapid than the decline to minimum, and they show variation of radial velocity with the period of light changes. The problem of these Variables has attracted much attention recently. It is supposed that the Cepheids are gaseous bodies expanding and contracting alternately and that such pulsations are the cause of the variation in the star's brightness. The dynamics of this Pulsation Theory is investigated in two papers on the subject by Eddington (See: *Monthly Notices* of the Royal Astronomical Society, *November* 1918 *and January* 1919).

Nizamiah Observatory,
Hyderabad, (*Deccan*).

T. P. Bhaskara Sastri.

SOLUTIONS.

Question 985

(M. K. Kewalramani) :—If A′,B′,C′ are points in the sides of the triangle A B C, such that B A′ : A′C = C B′ : B′A = AC′ : C′B = m : 1, find an expression for the radius of the circumcircle of the triangle D E F formed by the intersections of A A′, B B′, C C′.

Solutions by N. B. Pendse and N. B. Mitra.

The length of A A′ can be easily found, for

$$(m+1)\ AA'^2 + mA'C^2 + B\,A'^2 = m\,b^2 + c^2$$

i.e.
$$(m+1)\ A\,A'^2 = m\,b^2 + c^2 - m\,a^2.\ \frac{1}{(m+1)^2} + a^2\,\frac{m^2}{(m+1)^2}$$

$$= m\,b^2 + c^2 - \frac{m\,a^2}{m+1}.$$

$$\therefore \quad AA' = \frac{\sqrt{m\,(m+1)\,b^2 + (m+1)\,c^2 - ma^2}}{m+1}.$$

similarly
$$BB' = \frac{\sqrt{m\,(m+1)\,c^2 + (m+1)a^2 - mb^2}}{m+1},$$

and
$$CC' = \frac{\sqrt{m\,(m+1)\,a^2 + (m+1)b - m\,c^2}}{m+1},$$

The trilinear equation of B B′ is

$$a\,\alpha - mc\,\gamma = 0;$$

similarly CC′, AA′, are
$$b\,\beta - ma\,\alpha = 0,$$
$$c\,\gamma - m\,b\,\beta = 0.$$

The points B′, C′, A′ are given by

$$\frac{a\alpha}{m} = \frac{c\gamma}{1} = \frac{b\beta}{0} = \frac{2\Delta}{m+1},$$
$$\frac{b\beta}{m} = \frac{a\alpha}{1} = \frac{c\gamma}{0} = \frac{2\Delta}{m+1},$$
$$\frac{c\gamma}{m} = \frac{b\beta}{1} = \frac{a\alpha}{0} = \frac{2\Delta}{m+1},$$

Also, the points D, E, F are given by

$$\frac{a\alpha}{m} = \frac{b\beta}{m^2} = \frac{c\gamma}{1} = \frac{2\Delta}{m^2+m+1},$$
$$\frac{a\alpha}{1} = \frac{b\beta}{m} = \frac{c\gamma}{m^2} = \frac{2\Delta}{m^2+m+1},$$
$$\frac{a\alpha}{m^2} = \frac{b\beta}{1} = \frac{c\gamma}{m} = \frac{2\Delta}{m^2+m+1}.$$

$$\therefore \quad \text{area BDC} = \text{CEA} = \text{AFB} = \frac{\Delta m}{m^2+m+1}.$$

$$\therefore \quad \text{area DEF} = \frac{\Delta(1-m)^2}{m^2+m+1}.$$

And $$\frac{BD}{BB'}=\frac{m+1}{m^2+m+1},$$

$$\frac{BF}{BB'}=\frac{m(m+1)}{m^2+m+1}.$$

$\therefore$ $$\frac{FD}{BB'}=\frac{1-m^2}{m^2+m+1}.$$

Similarly $$\frac{EF}{AA'}=\frac{FD}{BB'}=\frac{DE}{CC}=\frac{1-m^2}{m^2+m+1}.$$

The circum radius of DEF is equal to

$$\frac{EF.\ FD.\ DE}{4 \text{ area } DEF}=\frac{BB'.\ AA'.\ CC'}{4\triangle}\times\frac{(1-m)^2}{m^2+m+1}\times\left(\frac{1-m^2}{m^2+m+1}\right)^2$$

$$=\frac{BB'.\ AA'.\ CC'}{4\triangle}\times\frac{(1-m)(1+m)^2}{(m^2+m+1)^2}$$

$$=\frac{1-m}{4\triangle(m^3+m+1)^2}\times\sqrt{\{(a^2+m^2c^2+2mac\cos B)}$$

$$(b^2+m^2a^2+2mab\cos C)\ (c^2+m^2b^2+2mbc\cos A)\}$$

Partial Solution by G. V. Krishnaswami and Sadanad.

Question 986.

(M. K. Kewalramani) :—At any point P of a curve equal arcs PQ, PR are measured in the same direction along the curve and along the parabola of the closest contact. Prove that ultimately

$$QR=\left\{3+\frac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}\right\}\frac{(ds)^4}{24\rho^3}.$$

Solution by Sadanand, N. B. Mitra, and S. R. Ranganathan.

Let the parabola be $y=(ax+b_1y)^2$. If it is of the closest contact with

$$y=\frac{s^2}{2\rho}-\frac{s^3}{6\rho^2}\frac{d\rho}{ds}-\frac{s^4}{24\rho^3}\left(1-2\left(\frac{d\rho}{ds}\right)^2+\rho\frac{d^2\rho}{ds^2}\right)+\ldots$$

$$x=s-\frac{s^3}{6\rho^2}+\frac{s^4}{8\rho^3}\frac{d\rho}{ds}+\ldots$$ [See: Edwards' *Calculus*, Ex. 51, P. 291];

then $$a^2=\frac{1}{2\rho},\quad ab=-\frac{1}{6\rho}\frac{d\rho}{ds}.$$

Substituting for x and y, we have

$$y=\frac{s^2}{2\rho}-\frac{1}{6\rho^2}\frac{d\rho}{ds}s^3+s^4\left(-\frac{1}{6\rho^3}+\frac{5}{72\rho^2}\left(\frac{d\rho}{ds}\right)^2\right)+\ldots$$

Hence ultimately,

$$QR=(y_1-y_2)$$

$$=\left(3+\frac{1}{3}\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}\right)^2\left(\frac{ds^4}{24\rho^3}\right).$$

Question 987.

(V. TIRUVENKATACHARI):—show that

$$\int_0^{\frac{1}{2}\pi} \sin\theta\ (\log\cos\theta)\ d\theta = -\frac{1}{3}\left\{1+\frac{2}{5}+\frac{2.4}{5.7}+\frac{2.4.6}{5.7.9}+\ldots\right\} = -1.$$

Solution by K. J. Sanjana and several others.

More generally let us take $\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2r+1} \log\cos\theta\ d\theta \equiv \mathrm{I}$.

Putting $\cos\theta = z$, we readily get $\mathrm{I} = \int_0^1 \log z.\ (1-z^2)^r dz$

$$= \int_0^1 \log z\ (1_1 - r_1 z^2 + r_2 z^4 \ldots + (-1)^r z^{2r})\ dz$$

$$(\alpha) \qquad = -\left\{\frac{r_0}{1^2} - \frac{r_1}{3^2} + \frac{r_2}{5^2} \ldots + (-1)^r \frac{1}{(2r+1)^2}\right\}.$$

Again, $$1 = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2r+1}.\ \log\ (1-\sin^2\theta)\ d\theta$$

$$= -\tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \{\sin\theta)^{2r+3} + \tfrac{1}{2}\ (\sin\theta)^{2r+5} + \tfrac{1}{3}\ (\sin\theta)^{2r+7} \ldots\}\ d\theta$$

$$= -\tfrac{1}{2}\left\{\frac{2\cdot4\ldots(2r+2)}{3\cdot5\ldots(2r+3)} + \tfrac{1}{2}\cdot\frac{2\cdot4\ldots(2r+4)}{3\cdot5\ldots(2r+5)} + \tfrac{1}{3}\cdot\frac{2\cdot4\ldots(2r+6)}{3\cdot5\ldots(2r+7)} + \ldots\right\}$$

$$(\beta) \qquad = -\tfrac{1}{2}\cdot\frac{2\cdot4\ldots(2r+2)}{3\cdot5\ldots(2r+3)}\left\{1+\tfrac{1}{2}\frac{2r+4}{2r+5} + \tfrac{1}{3}\frac{(2r+4)\ (2r+6)}{(2r+5)\ (2r+7)} + \ldots\right\}.$$

The result in the question is obtained by putting $r=0$ in these values. By equating (α) and (β) we get the algebraical identity—

$$\frac{r_0}{1^2} - \frac{r_1}{3^2} + \frac{r_2}{5^2} - \ldots + (-1)^r \frac{1}{(2r+1)^2}$$

$$= \frac{2\cdot4\ldots(2r)\ (r+1)}{3.5\ldots(2r+1)\ (2r+3}\left\{1+\tfrac{1}{2}\frac{2r+4}{2r+5} + \tfrac{1}{3}\frac{(2r+4)\ (2r+6)}{(2r+5)\ (2r+7)} + \ldots ad\ in\ f.\right\}.$$

Question 988.

(S. MALHARI RAO)—Complete the following magic square, by placing 8 other prime numbers in the vacant cells :—

7	43	71	113
73	97	23	41

Solution by Prof. N.B. Mitra, and several others.

We shall suppose that only positive primes are meant.

If we fill up the 1st two cells of the 3rd row by x & y, it is easy to see that the remaining cells should be filled up as follows :—

7	43	71	113
$154-x$	$94-y$	$48-y$	$x+2y-62$
x	y	$92+y$	$142-x-2y$
73	97	18	41

Hence $y<48$ and y, $94-y$ & $92+y$ are primes different from those given and from each other. The only value of y which satisfies these conditions is 11.

Hence the square now becomes

7	43	71	113
$154-x$	83	37	$x-40$
x	11	103	$120-x$
73	97	23	41

Hence $40<x<120$ and x, $x-40$, $120-x$, $154-x$ are all primes different from the others and from each other.

The only values of x satisfying these conditions are 53, 101 & 107.

Hence we get the following squares :—

7	43	71	113
47	83	37	67
107	11	103	13
73	97	23	41

7	43	71	113
53	83	37	61
101	11	103	19
73	97	23	41

7	43	71	113
101	83	37	13
53	11	103	67
73	97	23	41

Question 989.

(S. Malhari Rao) :—Give the solutions in positive integers of the equation

$$(x+y+z)^2 = x\,y\,z.$$

Solutions by K. J. Sanjana and N. B. Mitra.

Solving as a quadratic in z we obtain

$$z = \tfrac{1}{2}xy-(x+y) \pm \tfrac{1}{2}\sqrt{\{x^2y^2-4xy(x+y)\}}.$$

If $x\,y=u$, $x+y=v$, we require $u^2-4uv \equiv \square, = (u-2kv)^2$, say. This gives $u = v\,k^2/(k-1) = \alpha\,v$, suppose; $\therefore\ xy = \alpha x+\alpha y$, and $\dfrac{x}{\alpha} = \dfrac{y}{y-\alpha}$, so that $x = l\alpha$, $y = l\alpha/(l-1)$.

By giving k rational values and choosing l so as to make x and y integral, any number of solutions may be obtained. Thus $k=2$ along with $l=2$, 3, 5, gives the triads

8, 8, 16 ; 12, 6, 18 ; 20, 5, 25.

For this value of k, $u^2-4uv=0$. If $k=3$, $\alpha=\frac{9}{2}$; and $l=2$, $l=10$, give the pairs of triads 9, 9, 9 or 36 ; 45, 5, 100 or 25.

An easy practical method would be to put $y=ax$, $z=bx$; this gives $x=(1+a+b)^2/ab$. Values of a and b should now be so chosen as to make x integral : thus for

$a=1$, 1, 1, 2, 2, 3, 4, 5, 9, .., we get

$b=1$, 2, 4, 3, 9, 8, 5, 9, 20,...respectively ;
and the triads 9, 9, 9; 8, 8 , 16 ; 9, 9, 36 ; 6, 12, 18 ;...5, 45, 100 ;...may be found.

As the equation for finding x in terms of y and z is a quadratic, if one value of x for given values of y and z is known, a second value will always exist. Hence if this is x_1, we have

$$(x_1+\overline{a+b}\,x)^2 = a\,b\,x_1;$$

also
$$(x+\overline{a+b}\,x)^2 = a\,b\,x$$

$\therefore$
$$(x_1-x)\ \{x_1+x\,(1+2a+2b)\} = (x_1-x)\,a\,b\,x^2,$$

and $x_1 = x\ \{a\,b\,x-(1+2a+2b)\}$.

Thus starting with 6, 12, 18, and taking $x=6$, $x=2$, $b=3$, we get $x_1=150$, so that 12, 18, 150 is another triad of solutions. Next taking $x=12$, $a=\frac{3}{2}$, $b=\dfrac{25}{2}$, we get $x_1=2352$, and 18, 150, 2352 is another solution.

Similarly from this we shall get 150, 2352, 19321 ; and the process may be indefinitely continued.

Partial solutions by Sadanand, Pendse and V. Thiruvenkata Chari.

Questions for Solution.

1042. (MARTYN, M. THOMAS M.A.) :—The envelope of the curves $y \sin^2\theta = a \sin\theta - x \sin\theta \cos\theta - \frac{1}{c}(x - a\cos\theta)^2$, where θ is variable, consists of two parts. Give a dynamical interpretation of the result.

1043. (V. RAMASWAMI AIYAR, M.A.) :—On the sides BC, CA, AB of a triangle ABC pairs of points XX′, YY′, ZZ′ are taken such that XX′ = BC, YY′ = CA and ZZ′ = AB ; and perpendiculars are erected at these points to the sides. If the perpendiculars at (Y, Z′), (Z, X′), (X, Y′) intersect at P, Q, R ; and those at (Y′, Z), (Z′, X), (X′, Y) intersect at P′, Q′, R′ ; prove that the six points P, Q, R, P′, Q′, R′ all lie on a circle.

1044. (V. RAMASWAMI AIYAR, M.A.) :—Given three straight lines as well as their orthopoles : determine the circum-circle, as well as the nine points circle, of the triangle of reference. Show also how the triangle of reference could be determined.

1045. (N. P. PANDYA):—A variable parabola touches a fixed ellipse externally at P and the snb-normals at P to both the curves are equal. If a common tangent to the parabola and the auxiliary circle of the ellipse cuts the normal at P in Q, find the locus of Q.

1046. (S. KRISHNASWAMI AIYANGAR):—At every point O of a plane curve a parabola and a rectangular hyperbola of closest contact with it are drawn. If the common normal meets the conics at P and Q, then

$$9.QO^3 + 4\rho_1^2.PO = 0,$$

where ρ_1 is the radius of curvature at the corresponding point of the evolute to the curve.

1047. (S. KRISHNASWAMI AIYANGAR):—If ρ, be the radius of curvature at any point of a cardioide and ρ_o, ρ_α, $\rho_{\alpha\beta}$ at the corresponding points of its evolute, α—evolute, and the β—evolute of the α—evolute, show that

$$\rho_{\alpha\beta} = \rho_\alpha \sin\beta + \rho_o \sin\alpha \cos\beta - \frac{1}{9}\rho \cos\alpha \cos\beta.$$

1048. (V. THIRUVENKATACHARI, B.A. HONS) :—Establish the following asymptotic relations

$$\int_0^1 \frac{\sin\left(\log\frac{1}{y^p}\right)}{(1-xy)\log\left(\frac{1}{y}\right)}\, dy = \tan^{-1}\left(\frac{p}{1}\right) + x\tan^{-1}\left(\frac{p}{2}\right) + x^2\tan^{-1}\frac{p}{3} + \ldots\ldots$$

$$\int_0 \frac{\cos\left(2\log\frac{1}{y}\right)}{(1-xy)\sqrt{\log\frac{1}{y}}}\,dy = \sqrt{\pi}\left\{\frac{e^{-1}}{\sqrt{1}} + x,\ \frac{e^{-\frac{1}{2}}}{\sqrt{2}} + \frac{x^2 e^{-\frac{1}{3}}}{\sqrt{3}} + \frac{x^3 e^{-\frac{1}{4}}}{\sqrt{4}} + \ldots\ldots\right\}$$

when $|x| < k \leq 1$.

1049. (S. RAMANUJAN, F. R. S.) :—Show that

(1) $$\int_0^\infty \frac{\sin nx}{x + \frac{1}{x} + \frac{2}{x} + \frac{3}{x} + \ldots\ldots}\,dx = \frac{\sqrt{\left(\frac{1}{2}\pi\right)}}{n + \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots}$$

(2) $$\int_0^\infty \frac{\sin \frac{1}{2}\pi n x\,dx}{x + \frac{1^2}{x} + \frac{2^2}{x} + \frac{3^2}{x} + \ldots} = \frac{1}{n} + \frac{1^2}{n} + \frac{2^2}{n} + \frac{3^2}{n} + \ldots$$

1050. (K. S. SRINIVASACHARI, M.A., L.T.) :—If

$$(ax^2+bx+c)^{-1} = p_0 + p_1 x + p_2 x^2 + \ldots\ldots$$

find in terms of a, b, c, x the value of

$$p_0^2 + p_1^2 x^2 + p_2^2 x^4 + \ldots\ldots$$

1051. (A. C. L. WILKINSON) :—If ABCD is a cyclic quadrilateral, the points A, B, C, D occurring in this cyclic order, the sum of the reciprocals of the perpendiculars from A, C on their Simson lines with respect to the triangles BCD, ABD respectively is equal to the sum of the reciprocals of the perpendiculars from B, D on their Simson lines with respect to the triangles ACD, ABC respectively.

1052. (A. C. L. WILKINSON) :—If the circle of curvature at any point P of an ellipse be drawn and O be any point on the tangent at P, prove that the straight line joining the points of contact of the other two tangents from O to the ellipse and its circle of curvature, the other common tangent and the tangent at P are concurrent.

1053. (A. C. L. WILKINSON) :—Two circles are such that triangles can be inscribed in the one which are circumscribed to the other. If P, Q, R be the points of contact of such a triangle with the circle about which it is circumscribed, show that the nine-point-circle of P, Q, R is fixed and that the locus of the centres, K, of the inscribed or escribed circles of the triangle PQR is the curve $IK^2 = 2r.\,NK$, where N is the common nine point centre, I the centre and r the radius of the common circumscribed circle of the triangles PQR.

Periodicals received from 16th March to 15th May 1919 (with contents.)

(1) *Acta Mathematica*, Vol. 42, No. 1:—*Zur Theorie der linearen Gleichungen*,—E. Study. The classification of sets of points,—Eric. H. Neville. *Auszug aus einem Briefe des Herrn Landau an den Herausgeber*—Edmund Landau. *Bibliographie.*

(2) *The American Mathematical Monthly*, Vol. 25, No. 10. December 1918:--The content of a second course in Calculus—E. J. Moulton. A system of Algebraic and Transcendental Equations—G. N. Bauer and H. L. Slobin. Practical solution of Linear Equations—H. T. Burgess.

(3) *The American Mathematical Monthly*, Vol. 26, Nos. 1 & 2. January and February 1919: Leonardo of Pisa and his *Liber Quadratorum*,—R. S. McClenon. On the Form of the Power series for an Algebraic Function—E. J. Wilczynski. Concerning a method for finding a Particular Integral—A. B. Coble. Book Review: Who was the first Inventor of the Calculus—F. Cajori. Discussion: Concerning a new method for Tracing Cardioids,—W. F. Rigge. Collegiate Mathematics for war service: Statistics in relation to the War—Roxana H. Vivian. On certain constructions of Descriptive Geometry—G. Loria. Difference Quotients—T. P. Ballantine. A Theorem in the Geometry of the Triangle—J. W. Clawson. On the Elementary Problem of closure on an Equilateral Hyperbola—A. Emch. Remarks on a previous Article—N. Altshiller.

(4) *The Astrophysical Journal*, Vol. 49, No. 1, January 1919: On the Dark markings of the Sky, with a catalogue of 182 such objects—E. E. Barnard. Studies based on Colors and Magnitudes in Steller Clusters—Harlow Shapeley. The Infra-Red Arc Spectrum of Iron—H. M. Randall and E. F. Barker. Discussion of some evidence on the origin of Radiation in the Tube-Resistance Furnace—Arthur S. King. The Infra-Red Arc Spectra of Cobalt, Nickel, Manganese and Chromium—H. M. Randall and E. F. Barker.

(5) *Bulletin* of the *Calcutta Mathematical Society*, Vol. 9, No. 2 1917—18:—The late Sir Gooroo Dass Banerjee—Sudhansukumar Banerji. On the figures of Equilibrium of two Rotating Masses of Fluid for the Exponential Potential, Part I—Abanibhushan Datta, Fourier's Series and its Influence on some of the Developments of Mathematical Analysis—A. C. Bose. On the Numerical calculation on

the Roots of the Equations $P_n^m(u)=0$ and $\frac{d}{du}P_n^m(u)=0$, regarded of Equations in n—Bholanath Pal. On some New Therems in the Geometry of Masses—Sasindrachandra Dhar. On the Electric Resistance of a Conducting Spheroid with given Electrodes—Sitachandra Kar.

(6) *Mathematics Teacher*, Vol. 11, No. 2, December 1918:—Why Students Fail in Mathematics—H. A. Merrill. A Solution of Equations by Standard Curves—R. C. Colwell. War Problems in Mathematics—W. E. Breckenridge. Arithmetical Errors made by High School Pupils-T. H. Minnick. Some Relations connecting the sums of the Coaxial Minors of a Circulant—W. H. Metzler. Canada's Challenge: In Flanders Fields—T. D. McCrae. America's Answer—R. W. Lillard.

(7) *Monthly Notices of the Royal Astronomical Society*, Vol. 79, Nos. 2 & 3, December 1918 & January 1919:—The Evolution of Binary systems—J. H. Jeans. On a certain relation between the Elements a Planets and Satellites,—R. J. Pocock. Observations of the Magnitude of Nova Aquilae, 1918, made at the Radcliffe Observatory, Oxford; and a comparison with the Radcliffe Magnitudes of Nova Persei, 1901, A. A. Rambout. The earlier spectrum of Nova Aquilae, 1918,—A. L. Cortie. The dark-line spectrum of Nova Germinorum No. 2, J. Lunt. The Stellar Magnitude Scales of the Astrographic Catalogue. Thirteenth note. The Bordeaux Magnitudes ($+17^\circ$, $+16^c$, $+^\circ 15^c$ and $+14^\circ$)—H. H. Turner. Z. Andromedae (231848)—T. E. Espin. Some probably Variable Stars—T. E. Espin. *Sur les relations entre les coordonnees rectilignes normales d'une meme e'toile sur deux cliches differents.*—B. Bailland. On a shower of shooting stars from the Pleiades,—A. King. The New Star of Hipparchus, and the dates of Birth and Accession of Mithridates—J. K. Fortheringham. Obeservations of Nova Aquilae, 1918—J. G. Hagen. The Spectrum of Nova Aquilae, 1911, June 15—A. L. Cortie. The Pulsations of a Gaseous Star, and the Problem of the Cepheid Variables—A. S. Eddington. On the Variable Star B. D. $+39^\circ 3476$ = Piazzi XVIII, 153—A. Stanley Williams. The Short-period Variable R. R. Ceti, C. Martin and H. C. Plummer. Magnitude Observations of five Wolf-Rayet Stars—C. Martin and H. C. Plummer. Proper motions of the stars in Zone $24^\circ - 32^\circ$ N. declination, in relation to Spectral Types—F. W. Dyson and W. G. Thackeray. Occultations of stars by the Moon and Phenomena of Jupiter's Satellites observed in year 1918—Royal Observatory, Greenwich. Mycrometrical measures of Double Stars (13th Series) and New Double Stars—T. E. Espin. Observations of a bright projection on the limb of Jupiter—F. Sargent.

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MR. S. RAMANUJAN, B. A., F. R. S.

THE JOURNAL
OF THE
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Vol. XI.] AUGUST 1919. [No. 4.

PROGRESS REPORT.

In continuation of the announcement in the April number of our Journal, the Committee feel very great pleasure to inform the General Body that under Art. VII (*b*) of the Constitution, Mr. S. Ramanujan B.A., F.R.S. has been appointed *an Honorary Member* of our Society. A short life-sketch of Mr. Ramanujan is given elsewhere in this number and a portrait, prepared with as much care as possible from an old group-photo, adorns this number as its frontispiece.

2. The following gentlemen have been elected members of our Society :—

1. *Mr. Mohanlal V. Vasawada, L.C.E.*—Assistant Member; Board of Revenue for Irrigation, Gwaliar State, Morar (C.I.)
2. *Mr. Harihar Nath Wancha, M.A.*—Professor of Mathematics Rangoon College, Rangoon ;
3. *Mr. Nripendranath Maitra, M.A., B.L.* (Cal.),—Deputy, Magistrate and Deputy Collector, Sitamarhi (P.C.), (Dist.—Muzaffarpur, Behar) ;
4. *Mr. Abani Kanta Ray, B. Sc.*,—M. Sc. Student, Cal. Univ. College, Krishnagar (Bengal). (at concessional rate) ;
5. *Mr. R. Sriramulu*—Lecturer, Pachaiyappas College, Madras ;
6. *Mr. K. Narasimha Rao*—Assistant Lecturer, Madras Christian College, Madras ;
7. Mr. *A. Nageshwara Aiyar, B.E.*—District Board Engineer, Ramnad.

3. The Committee feel it their very sad duty to record the premature death of His Highness the Maharaja Bhowsinghji of Bhownagar. The late Maharaja was a highly enlightened Ruling Prince and had cheerfully got himself enrolled as a Life-Member of our Society in its very early existence. By his untimely death our Society has sustained a heavy loss.

4. The following books have been presented to our Library—

1. *Analytical Geometry* of the Straight Line and the Circle-by John Milne, George Bell & Sons, 5s/-, 1919 ;
2. *Differential Calculus* for Colleges and Secondary Schools-by Charles Davison, G. Bell & sons, 6s/-, 1919 ;
3. *An Arithmetic* for Preparatory schools-by Trevor Dennis, G. Bell & Sons. 4s/6d, 1919 ;
4. *Bomboy University Calendar*, Parts I & II for 1919-20 ;
5. *Madras University Calendar*, Parts I & II for 1919, and Exam. Papers for 1918.

POONA,
10th *Aug.* 1919.

D. D. KAPADIA,
Honorary Joint Secretary.

Mr. S. Ramanujan, B.A., F.R.S.

Mr. S. Ramanujam B.A. F.R.S., a Sri Vaishnava Brahmin was born of poor parents in December 1888 at Erode in the Madras Presidency. He was put to school when he was five years old and even while in the Lower Secondary Forms he gave indication of his special bent of mind for mathematics by questioning his teachers about zero and imaginary quantities. He matriculated from the Town High School, Kumbhakonam in December 1903 and read for about a year in the F. A. Class in the Government College, Kumbhakonam. Even at that stage he was working at infinite series and different representations of numbers.

Having failed in the F. A. Examination, a natural result of his absorption in mathematics and consequent neglect of other subjects, he spent nearly 5 years moving about from place to place andworking at mathematics in his own way. In 1910, he went to Madras with two good sized note books filled with his researches in mathematics and with the help of some friends obtained a clerk's post in the Harbour Trust ;Office. This enabled him to remain in Madras and work at mathematics with the help of books and periodicals which were made accessible to him there. In 1912, having one day seen the remark in Mr. G. H. Hardy's tract on 'Orders of Infinity' that the precise order of $\rho(x)$ had not yet been ascertained and having himself arrived at a result relating to it, he opened correspondence with Mr. Hardy sending him some of his results on continued fractions and theory of numbers. Mr. Hardy was struck with the grandeur of the results and wrote to him an appreciative letter asking for more of his results in other branches of mathematics. On receiving these results which were mainly in Definite Integrals and Elliptic Functions. Mr. Hardy discovered in Ramannjan a great Mathematician and asked him if he would go to Cambridge. Meanwhile the Madras University recognising his Mathmatical talents had granted him a Research Studentship. Mr. E. H. Neville of Cambridge who came to Madras to deliver a course of University Lectures and who had been asked by Mr. Hardy to bring Mr. Ramanujan, with him when he returned to Cambridge, got the University of Madras to grant Mr. Ramanujan a special Scholarship of the annualvalue of £. 250 for three years, and Mr. Ramannjan sailed for England in March 1914. He was very warmly received at Cambridge by Mr. Hardy and both he and Mr. Littlewood helped him in publishing his researches in the English Periodicals. An account of his work while in England has already appeared in the February 1917 number of this Journal. At Cambridge he was given the Research Degree and the frontispiece shows him in his accademical robes. In recognition of his abilities in Mathematics, he was elected a Fellow of the Royal Society in 1917, and in 1918 he was granted a Fellowship of Trinity. The Madras University has also been pleased to give him an annual grant of £. 250 for another period of 5 years without imposing any conditions regarding residence or work. Since May 1917, he has not been keeping good health. He returned to India in March last for the sake of his health and is now residing at Kodumudi a village on the banks of the Cauvery and very near the place of his birth.

On the Cartesian Oval.

By V. Ramaswami Aiyar.

Introduction.

In tackling a problem connected with Genocchi's theorem on the rectification of the cartesian oval, I found that a method of treating the curve which dealt with the three foci symmetrically, and furnished simple expressions for the leading magnitudes connected with the oval, was a desideratum. Then I groped my way to the discovery of the fundamental parametric equations here given. By means of them the treatment becomes simple and of the kind desired ; and, while the leading properties of the curve are deduced easily and in various ways, several new results occur, some of them quite of importance. I have not, so far, found the method here noticed anticipated.

In this article I shall be able to present only a part of the work that I wish to ; I hope to present the rest subsequently.

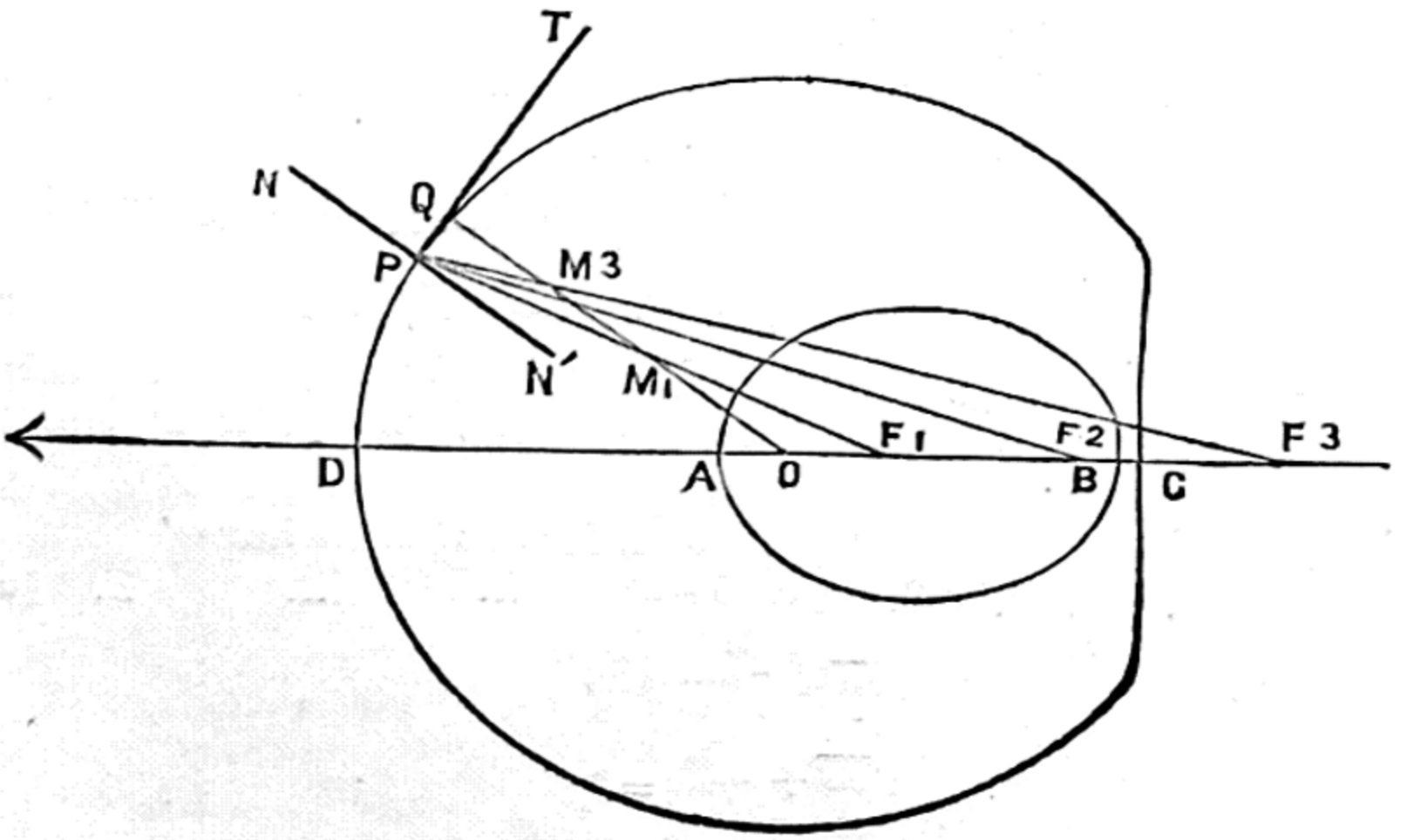
Section 1.

The Fundamental Parametric Equations.

2. A, B, C, being three given collinear points, the distances of any point P from them are connected by the well known relation

$$PA^2.BC+PB^2.CA+PC^2.AB+BC.CA.AB=0;$$

which will be referred to as the 'equation of distances.' By means of it, when any two of these distances are known, the third can be found.



3. Let now O, F_1, F_2, F_3 be four points in a line in order, we shall say, from left to right. We will call the line the axis. Let $OF_1=a$, $OF_2=b$, $OF_3=c$; so that $o<a<b<c$. Then a system of consistent values for the distances of a point P from F_1, F_2, F_3 are furnished by the equations

$$\left.\begin{aligned}\rho_1&=\sqrt{bc}+\frac{\lambda}{\sqrt{bc}}\\ \rho_2&=\sqrt{ca}+\frac{\lambda}{\sqrt{ca}}\\ \rho_3&=\sqrt{ab}+\frac{\lambda}{\sqrt{ab}}\end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

as may be verified at once by the equation of distances.

In these expressions there is a *parameter* λ. For every value of λ we shall have a point P, or rather a pair of points P, P′ situated symmetrically about the axis. They are the points of co-intersection of circles of centres F_1, F_2, F_3, and of radii ρ_1, ρ_2, ρ_3 corresponding to that value of λ. We shall speak of these circles as the 'circles' ρ_1, ρ_2, ρ_3. They are coaxal for every value of λ.

Varying λ, from $+\infty$ to $-\infty$, we shall have a *locus* of the points ρ, ρ'. We will call this locus 'the Cartesian Oval.' O will be called its 'triple focus'; and F_1, F_2, F_3 its 'foci' or 'single foci',

Eliminating λ between the equations (1) taken in pairs, we get the following 'bipolar' equations of the curve:

$$\left.\begin{aligned}\sqrt{b}\cdot\rho_1-\sqrt{a}\cdot\rho_2&=\sqrt{c}.\ (b-a)\\ \sqrt{c}\cdot\rho_1-\sqrt{a}\cdot\rho_3&=\sqrt{b}.\ (c-a)\\ \sqrt{c}\cdot\rho_2-\sqrt{b}\cdot\rho_3&=\sqrt{a}.\ (c-b)\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Also using all the equations in (1), we get the following 'tripolar' equation of the curve.

$$\sqrt{a}.\ (b-c).\ \rho_1+\sqrt{b}.\ (c-a).\ \rho_2+\sqrt{c}.\ (a-b).\ \rho_3=0 \quad \ldots \quad (3)$$

4. **Theorem.** *If the distances of a point P from F_1 and F_2, taken with suitable signs, satisfy the first equation in* (2), *the point P must be on the oval.*

For, let the distances so taken be δ_1, δ_2: so that we have

$$\sqrt{b}\cdot\delta_1-\sqrt{a}\cdot\delta_2=\sqrt{c}.\ (b-a) \quad \ldots \quad \ldots \quad \ldots \quad (\alpha)$$

Putting

$$\sqrt{bc}+\frac{\lambda}{\sqrt{bc}}=\delta_1,$$

$$\sqrt{ca}+\frac{\lambda}{\sqrt{ca}}=\delta_2,$$

and solving for λ, we get the *same* value of λ from each equation. This is by virtue of the relation (α). This proves that P is the poin on the curve whose parameter is λ so determined.

It is clear that a similar statement holds with regard to the other equations in (2).

Hence, if the symbol (AB) mean 'the distance AB taken with a suitable sign,' we see that our oval is the locus of a point P moving so that any of the following equations is satisfied :

$$\left.\begin{aligned} \sqrt{b}.\ (PF_1)+\sqrt{a}.\ (PF_2)&=\sqrt{c}.\ F_1F_2\ ;\\ \sqrt{c}.\ (PF_1)+\sqrt{a}.\ (PF_3)&=\sqrt{b}.\ F_1F_3\ ;\\ \sqrt{c}.\ (PF_2)+\sqrt{b}.\ (PF_3)&=\sqrt{a}.\ F_2F_3\ ; \end{aligned}\right\} \quad \ldots \quad \ldots \quad (4)$$

The Ordinary Form Identified.

5. Let now A, B, be two given points; l, m, n any positive, unequal, multiples; and let a variable point P move so that

$$m\ (PA)+l\ (PB)=n.\ AB. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Then according as n is the greatest, intermediate, or least of the quantities l, m, n, the equation becomes comparable with the first, second, or third equation in (4) ; and we conclude :

(i) that the locus of P is a Cartesian Oval having A and B for two of its foci ;

(ii) that the triple focus O is a point situated in AB produced such that $\frac{OA}{l^2}=\frac{OB}{m^2}$;

(iii) that it has for the third focus a point C situated in the line on the same side of O as A and B, such that

$$\frac{OA}{l^2}=\frac{OB}{m^2}=\frac{OC}{n^2}\ ;$$

(iv) and that the point P also satisfies the equations

$$n\ (PA)+l\ (PC)=m.\ AC,$$
$$n\ (PB)+m(PC)=l.\ BC,$$

Having made this identification we shall leave the ordinary form and proceed to discuss the curve by means of the special equations.

Section II.

The Vertices of the Oval and its General Tracing.

6. By the vertices of the oval we shall mean the points where the curve cuts the axis. Let x be the abscissa and λ the parameter of a vertex. We shall measure the abscissa from O as origin, and positively

to the left. The foci then are the points with abscissae $-a$, $-b$, and $-c$; and $x+a$, $x+b$, $x+c$ are equal to the focal distances of the vertex numerically. So also are ρ_1, ρ_2, ρ_3 for the value of λ in question.

Hence x and λ must satisfy one equation out of each of the pairs:

$$x+a=\pm\rho_1\ ;\ x+b=+\rho_2\ ;\ x+c=\pm\rho_3.$$

We find the equations go together as follows:

(1) $x+a=\rho_1,\ x+b=\rho_2,\ x+c=\rho_3$;

(2) $x+a=\rho_1,\ x+b=-\rho_2,\ x+c=-\rho_3$;

(3) $x+a=-\rho_1,\ x+b=\rho_2,\ x+c=-\rho_3$;

(4) $x+a=-\rho_1,\ x+b=-\rho_2,\ x+b=\rho_3$.

We may also observe that the equations (2) are what (1) become on changing $\sqrt{a}$ into $-\sqrt{a}$; and (3) and (4) are similarly what (1) become on changing the signs of $\sqrt{b}$ and $\sqrt{c}$.

Denoting the values of x and λ given by these equations in order by

$$(x_4,\ \lambda_4)\ (x_1,\ \lambda_1)\ (x_2,\ \lambda_2)\ (x_3,\ \lambda_3)$$

we find that they are

$$\begin{aligned}
x_4&=\sqrt{bc}+\sqrt{ca}+\sqrt{ab},\ \lambda_4=\sqrt{abc}\ (\sqrt{a}+\sqrt{b}+\sqrt{c}),\\
x_1&=\sqrt{bc}-\sqrt{ca}-\sqrt{ab},\ \lambda_1=\sqrt{abc}\ (\sqrt{a}-\sqrt{b}-\sqrt{c}).\\
x_2&=-\sqrt{bc}+\sqrt{ca}-\sqrt{ab},\ \lambda_2=\sqrt{abc}\ (-\sqrt{a}+\sqrt{b}-\sqrt{c})\\
x_3&=-\sqrt{bc}-\sqrt{ca}+\sqrt{ab},\ \lambda_3=\sqrt{abc}\ (-\sqrt{a}-\sqrt{b}+\sqrt{c}).
\end{aligned}\qquad \ldots\ (6)$$

Now on the axis set off the points of abscissae x_4, x_1, x_2, x_3 as above given; and call them D, A, B, C. These are the vertices of the curve; and they are the points of parameters λ_4, λ_1, λ_2, λ_3.

It should be observed that the vertices and the foci lie along the axis in the order

$$\mathrm{D},\ \mathrm{A},\ \mathrm{F}_1,\ \mathrm{F}_2,\ \mathrm{B},\ \mathrm{C},\ \mathrm{F}_3.$$

7. For tracing the curve it is necessary to note the relative order of the values of λ which correspond to the vertices *and* of those which make ρ_1, ρ_2, ρ_3 respectively vanish, namely,

$$\lambda'=-bc,\ \lambda''=-ca,\ \lambda'''=-ab.$$

It is readily seen that they are in descending order of magnitude as follows:

$$\lambda_4,\ \lambda_3,\ \lambda''',\ \lambda'',\ \lambda_2,\ \lambda_1,\ \lambda'.$$

Also we note, from the fundamental parametric equations, that the increments (or decrements) in the values of ρ_1, ρ_2, ρ_3 corresponding to any given change in the value of λ are of the forms

$$\sqrt{a}.\delta,\ \sqrt{b}.\delta,\ \sqrt{c}.\delta.$$

(1) As λ decreases from ∞ to λ_4, the circles ρ_1, ρ_2, ρ_3 decrease in radius and come into contact at the vertex D. The values of ρ_1, ρ_2, ρ_3 corresponding to $\lambda=\lambda_4$ are F_1D, F_2D, F_3D. Hence prior to λ reaching the value λ_4 the radii of the circles ρ_1, ρ_2, ρ_3, have been of the forms $F_1D+\sqrt{a}.\delta$, $F_2D+\sqrt{b}.\delta$, $F_3D+\sqrt{c}.\delta$; and, plotting these for a typical δ, we see that the circle ρ_1 is entirely within the circle ρ_2, and the latter within the circle ρ_3. Our tracing points ρ,ρ' have thus been imaginary until the value λ_4 was reached, when, they both appear at the position D.

(2) As λ decreases from λ_4 to λ_3, the radii of the circles ρ_1, ρ_2, ρ_3, are to start with, of the forms

$$F_1D-\sqrt{a}.\delta,\ F_2D-\sqrt{b}.\delta,\ F_3D-\sqrt{c}.\delta.$$

Plotting these for a typical δ, it is at once seen that the circles ρ_1, ρ_2, ρ_3 interesect one another. Thus as λ decreases from λ_4, the coaxal circles ρ_1, ρ_2, ρ_3, form an intersecting system; and this must continue until the next contact occurs, that is, until the value λ_3 is reached. Then the circles touch one another at the vertex C, and the values of ρ_1, ρ_2, ρ_3, there are F_1C, F_2C, F_3C. Thus, as λ decreases from λ_4 to λ_3 our tracing points P,P′ are real and trace out an oval extending from D to C on both sides of the axis.

(3) As λ decreases from λ_3 to λ_2, the radii of the circles are, to start with, of the forms $F_1C-\sqrt{a}.\delta$. $F_2C-\sqrt{b}.\delta$, $F_3C-\sqrt{c}.\delta$. Plotting these for a small δ, it is seen that the circle ρ_2 becomes entirely contained in the circle ρ_1, while the circle ρ_3 goes entirely outside the circle ρ_1. This kind of *non-intersection* goes on until the next contact is reached, that is until $\lambda=\lambda_2$. In the interval, the radius ρ_3 diminished to zero when λ passed through the value "$\lambda'=ab$, and then became negative numerically increasing. The like happened with the radius ρ_2 as λ passed through the value $\lambda''=-ca$. When λ becomes equal to λ_2, the circles P_1, P_2, P_3 come into contact at B, the value of the radii then reached being F_1B, $-F_2B$, and $-F_3B$ respectively. The tracing points P, P′ have been imaginary in the interval.

(4) As λ decreases from λ_2 to λ_1, the radii of the circles are, to start with, of the forms

$$F_1 B - \sqrt{a}.\delta, \quad -F_2 B - \sqrt{b}.\delta, \quad -F_3 B - \sqrt{c}.\delta.$$

Plotting these typically, it is at once seen that the circles ρ_1, ρ_2, ρ_3 start in a state of real intersection and the tracing points P, P′ are real. This goes on until the next contact is reached, that is till $\lambda =$ when the radii of the circles become F_1A, $-F_2A_2$ and $-F_3A$. Thus as λ decrease from λ_2 to λ_1 our tracing points are real and trace out an oval extending from B to A on both sides of the axis.

(5) In like manner it is seen that as λ decreases from λ_1 to -8, our tracing points are imaginary. The circle ρ_1 is entirely contained in the circle ρ_2 and the latter in the circle ρ_3; and this goes on however large λ becomes negatively.

8. Summarising, we see that the real part of the curve consists of two ovals—together called the Cartesian Oval—one extending from D to C and traced as λ ranges from λ_4 to λ_3 and the other from B to A and traced as λ ranges for λ_2 to λ_1.

The latter oval is entirely contained in the former; for ρ_1 has throughout the range λ_4 to λ_1 been positive and decreasing. The two constituent ovals are thus distinguishable as the *outer* and the *inner* oval.

It is to be particularly noted that the values of ρ_1, ρ_2, ρ_3 are positive for points on the outer oval; while *for points on the inner oval, ρ_1 is positive, but ρ_2 and ρ_3 are negative*. The equations (2) are therefore true in an *arithmetical* sense for the outer oval; and they would be true in the same sense for the inner oval upon changing the signs of ρ_2 and ρ_3.

We further observe that the foci F_1 and F_2 are contained within both the ovals, while F_3 is outside both. Hence F_3 is termed the *external* focus. Also, the triple focus O will be within, on, or outside the inner oval according as the abscissa of the vertex A is positive, zero, or negative; that is according as

$$\sqrt{bc} \gtrless \sqrt{ca} + \sqrt{ab}. \qquad \ldots \qquad \ldots \quad (7)$$

Section III.

The fourth Parametric Equation.

9. Let P be a point on the oval of parameter λ. The distances of P from F_1, F_2, F_3 being known, its distance from any other point on

the axis can be found by means of the equation of distances. Let ρ denote the distance to the triple focus O. We find

$$\rho^2 = bc + ca + ab + 2\lambda_2 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

10. Eliminating λ between the values of ρ^2 and of ρ_1, ρ_2, ρ_3 in succession, we get the bipolar equations

$$\left.\begin{aligned} \rho^2 &= 2\sqrt{bc}.\,\rho_1 + ab + ac - bc \\ \rho^2 &= 2\sqrt{ca}.\,\rho_2 + bc + ba - ca \\ \rho^2 &= 2\sqrt{ab}.\,\rho_3 + ca + cb - ab \end{aligned}\right\} \quad \ldots \quad \ldots \quad (9)$$

11. Let $\Theta_1, \Theta_2, \Theta_3$ be the angles that F_1P, F_2P, F_3P make with the axis, so that (ρ_1, Θ_1), (ρ_2, Θ_2), (ρ_3, Θ_3) are the polar co-ordinates of P with respect to F_1, F_2, F_3 as origins, the initial line in each case being the axis pointing *to the left*. We have

$$\left.\begin{aligned} \rho^2 &= \rho_1^2 - 2a\rho_1 \cos\Theta_1 + a^2 \\ \rho^2 &= \rho_2^2 - 2b\rho_2 \cos\Theta_2 + b^2 \\ \rho^2 &= \rho_3^2 - 2c\rho_3 \cos\Theta_3 + c. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (10)$$

12. Using these values in (9), we get the following polar equations of the oval :

$$\left.\begin{aligned} \rho_1^2 - 2\rho_1(\sqrt{bc} + a\cos\Theta_1) + (a-b)(a-c) &= 0 \\ \rho^2 - 2\rho_2(\sqrt{ca} + b\cos\Theta_2) + (b-a)(b-c) &= 0 \\ \rho_3^2 - 2\rho_3(\sqrt{ab} + c\cos\Theta_3) + (c-a)(c-b) &= 0. \end{aligned}\right\} \quad \ldots \quad (11)$$

The polar equations make it evident that the curve is the inverse of itself with respect to the circles of centres F_1, F_2, F_3 and squares of radii equal to $(a-b)(a-c)$, $(b-a)(b-c)$, $(c-a)(c-b)$. The first and third are real circles, while the second is imaginary. We shall speak of them as the circles F_1, F_2, F_3. Any two of the foci are inverses with respect to the circle whose centre is the third. The circles F_1 and F_2 invert each constituent oval into the other; while the circle F_3 inverts each into itself.

The circle F_3 cuts each constituent oval into two parts one of which lies without, and the other within, the circle F_3. They may be distinguished accordingly as the *outer* and *inner* parts of that oval. the terminal points of the parts are the points of contact of the tangents from F_3.

We shall generally take our typical point P of the curve to be on the outer part of the outer oval. Theorems proved for the case will be found to hold when P is elsewhere with suitable modifications suggested by continuity.

13. In our curve let us allow a to increase and become equal to b. Then the first and second equation in (11) become

$$r=2(\sqrt{bc}+b\cos\theta), \quad b<c \qquad \ldots \quad (12)$$

an elliptic limacon.

Next, let c decrease and become equal to b. Then the second and third equations in (11) become

$$r=2(\sqrt{ab}+b\cos\theta,) \; b>a \qquad \ldots \quad (13)$$

a hyperbolic limacon.

Thirdly, let a, b, c all become equal. Then each equation becomes

$$r=2a\,(1+\cos\theta), \qquad \ldots \quad (14)$$

a cardioid. Thus the limacons and the cardioid are extreme cases of the cartesian oval.

Lastly, let O advance infinitely near to, and become coincident with, F_1. The outer and inner ovals coalesce and the curve becomes the circle F_1 reckoned twice over. This also is an extreme form of the oval.

14. Before passing to consider the cartesian equations of the curve we may take notice of a special case of the equations (9). Let a, b, c be such that $ab+ac-bc=0$. Then the first equation in (9) becomes

$$\rho^2=2\sqrt{bc}\;\rho_1. \qquad \ldots \qquad \ldots \qquad \ldots \quad (15)$$

This equation, then, represents a cartesian oval. It is readily seen that it inverts with respect to O into the bipolar form $rr'=k^2$, that is a Cassinian Oval. The condition $ab+ac-bc=0$ is equivalent to $a^2=(a-b)(a-c)$; or the circle of inversion F_1 passes through the triple focus O. Hence we have the theorem :

If in a Cartesian Oval the circle of inversion F_1 passes through the triple focus O, the curve inverts with respect to O into an oval of Cassini. (16)

15. The equations (9) are equally suitable for deducing the cartesian equations of the curve. If x, y be the co-ordinates of P, abscissas being counted positive to the left as before, we have

$$\rho^2=x^2+y^2, \; \rho_1^2=(x+a)^2+y^2, \text{ \&c.}$$

Hence we get

$$\left.\begin{aligned}(x^2+y^2-ab-ac+bc)^2&=4bc\,\{(x+a)^2+y^2\}\\(x^2+y^2-bc-ba+ca)^2&=4ca\,\{(x+b)^2+y^2\}\\(x^2+y^2-ca-cb+ab)^2&=4ab\,\{(x+c)^2+y^2\}.\end{aligned}\right\} \qquad \ldots \quad (17)$$

Each of these reduces to

$$(x^2+y^2-bc-ca-ab)^2-4abc\ (2x+a+b+c)=0, \quad \ldots \quad (18)$$

which may be considered the standard form of the cartesian equation of the oval.

16. When an equation is of the form $x^2=y^2+z^2$, the analytical geometer knows it at once as the envelope of the system of curves

$$x=y\cos\theta+z\sin\theta \quad \ldots \quad \ldots \quad (19)$$

where θ is a parameter.

Accordingly, from equations (17) we see that the curve is the envelope of each of the systems of circles

$$\left.\begin{aligned} x^2+y^2-ab-ac+bc&=2\sqrt{bc}\ \{(x+a)\cos\theta+y\sin\theta\}\ ;\\ x^2+y^2-bc-ba+ca&=2\sqrt{ca}\ \{(x+b)\cos\theta+y\sin\theta\}\ ;\\ x^2+y^2-ca-cb+ab&=2\sqrt{ab}\ \{(x+c)\cos\theta+y\sin\theta\}\ .\end{aligned}\right\} \quad \ldots \quad (20)$$

Written in their standard forms, these circles are

$$(x-\sqrt{bc}\cos\theta)^2+(y-\sqrt{bc}\sin\theta)^2=a(b+c+2\sqrt{bc}\cos\theta)\ ;\ \&c. \quad (21)$$

Let G_1, G_2, G_3 denote the circles of centre O and radii $\sqrt{bc}$, $\sqrt{ca}$, $\sqrt{ab}$ respectively. They are of importance in connection with the oval. We shall call them the 'generating circles' of the oval. From equations (21), it can be readily seen that

The Cartesian Oval is the envelope of a variable circle whose centre moves on any of the generating circles G_1, G_2, G_3, and which cuts the corresponding circle of inversion F_1, F_2 F_3, orthogonally.

These will be spoken of as the *non-axial*, or *secondary*, systems of bitangential circles of the oval.

From equations (18) we see that the cartesian oval is the envelope of the system of circles

$$x^2+y^2-bc-ca-ab=u(2x+a+b+c)+\frac{abc}{u}\ ; \quad \ldots \quad (22)$$

that is, of the system

$$(x-u)^2+y^2=\frac{(u+a)(u+b)u+c)}{u} \quad \ldots \quad (23)$$

which forms the equation of the *axial*, or *primary*, system of bitangential circles of the oval. Having thus introduced the analytical treatment of the curve, we proceed in the next section to develop a different line of study, also based on the fourth parametric equation, using Calculus.

Section IV.

The Triple Focal Perpendicular on the Tangent.

It is desirable to have a clear idea at the outset as to how the normal at a point P lies with reference to the joins of P to O, F_1, F_2, F_3.

Let P describe the outer oval from D to C. The distances of P to O, F_1, F_2, F_3 are

$$\rho=\sqrt{bc+ca+ab+2\lambda},\ \rho_1=\sqrt{bc}+\lambda/\sqrt{bc},\ \text{etc},$$

and they are all positive and decreasing. If PT be the tangent at P drawn in the direction of motion it follows that PO, PF_1, PF_2, PF_3 make acute angles with PT. *Hence the normal at* P *being drawn, the lines* PO, PF_1, PF_2, PF_3 *all lie on one side of it, namely the side of* PT.

Next let P describe the inner oval from B to A. Then the distances ρ and ρ_1 are positive and diminishing; while ρ_2 and ρ_3 are, *in magnitude,* increasing. Hence PO and PF_1 make acute angles with the tangent PT drawn in the direction of movement; while PF_2 and PF_3 make obtuse angles with the same. Hence the normal at P being drawn as before, we see PO and PF_1 will lie on one side of it, namely that of PT; while PF_2 and PF_3 will lie on the other. Therefore, *at any point* P *on the inner oval the normal lies within the angle* F_1PF_2.

The above will help us to draw the figure correctly for what follows.

19. Let P be a point on the curve, of parameter λ. Join it to O, F_1, F_2, F_3. Draw OQ perpendicular to the tangent at P, cutting PF_1, PF_2, PF_3 at M_1 M_2, M_3, and the tangent at P in Q. Then we get a figure with a few—I would say *five*—leading properties, from which numerous others follow. We shall deal with the first two of them in the present section.

20. *The first property is that*

$$PM_1=\sqrt{bc},\ PM_2=\sqrt{ca},\ PM_3=\sqrt{ab}. \qquad \ldots \quad (24)$$

Dem. Take the fundamental equations

$$\rho^2=bc+ca+ab+2\lambda,$$

$$\rho_1=\sqrt{bc}+\frac{\lambda}{\sqrt{bc}},$$

$$\rho_2=\sqrt{ca}+\frac{\lambda}{\sqrt{ca}},$$

$$\rho_3=\sqrt{ab}+\frac{\lambda}{\sqrt{ab}},$$

Differentiating with respect to the arc, measured from C towards D in the outer oval, and from A towards B in the inner oval, so that λ and s increase together, we get

$$\rho\frac{d\rho}{ds}=\frac{d\lambda}{ds},$$

$$\frac{d\rho_1}{ds}=\frac{1}{\sqrt{bc}}\cdot\frac{d\lambda}{ds}, \text{ etc.}$$

The first of these means $\rho \cos \mathrm{OPQ}=\frac{d\lambda}{ds}$;

that is $\frac{d\lambda}{ds}= t$, where t denotes PQ. (25)

Hence the second equation tells us

$$\cos \mathrm{F_1PQ}=\frac{t}{\sqrt{bc}};$$

that is

$$\frac{t}{\mathrm{PM_1}}=\frac{t}{\sqrt{bc}}.$$

Hence $\mathrm{PM_1}=\sqrt{(bc)}$; and similarly $\mathrm{PM_2}=\sqrt{(ca)}$, and $\mathrm{PM_3}=\sqrt{(ab.)}$.

Cor. 1. As the whole lengths $\mathrm{PF_1}$, $\mathrm{PF_2}$, $\mathrm{PF_3}$ are $\sqrt{(bc)}+\lambda/\sqrt{(bc)}$, etc.; and the parts $\mathrm{PM_1}$, etc., are equal to $\sqrt{(bc)}$, etc., therefore the remaining parts $\mathrm{M_1F_1}$ $\mathrm{M_2F_2}$ $\mathrm{M_3F_3}$ are equal respectively to

$$\frac{\lambda}{\sqrt{(bc)}}, \quad \frac{\lambda}{\sqrt{(ca)}}; \quad \frac{\lambda}{\sqrt{(ab)}}. \qquad \text{... (26)}$$

Cor. 2. Hence rectangle $\mathrm{PM_1}\cdot \mathrm{M_1F_1}$ = rectangle $\mathrm{PM_2}\cdot \mathrm{M_2F_2}$ = rectangle $\mathrm{PM_3}\cdot \mathrm{M_3F_3}=\lambda$. (27)

We are thus afforded a simple geometrical meaning of the parameter λ.

Cor. 3. The sines of the angles that the normal makes with the focal vectors $\mathrm{PF_1}$, $\mathrm{PF_2}$, $\mathrm{PF_3}$ are as $\sqrt{a}:\sqrt{b}:\sqrt{c}$.

For, these angles are equal to the angles $\mathrm{PM_1Q}$, $\mathrm{PM_2Q}$, $\mathrm{PM_3Q}$ in our figure; whose sines are $\frac{t}{\sqrt{(bc)}}$, $\frac{t}{\sqrt{(ca)}}$, $\frac{t}{\sqrt{(ab)}}$; and are therefore as $\sqrt{a}:\sqrt{b}:\sqrt{c}$.

21. **Theorem**: *If the normal at P cut the axis in G, then* $OG=abc/\lambda$. (28)

Dem: The normal is parallel to OM_1. Therefore by similar triangles

$$GO : OF_1 = PM_1 : M_1F_1 ;$$

that is,
$$OG : a = \sqrt{bc} : \frac{\lambda}{\sqrt{bc}} ;$$

$$\therefore \quad OG = \frac{abc}{\lambda}.$$

Cor. 1. The centres of curvature at the vertices D, A, B, C of the oval are the points of abscissae ξ_4, ξ_1, ξ_2, ξ_3 where

$$\left.\begin{aligned} \xi_4 &= \frac{\sqrt{abc}}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \\ \xi_1 &= \frac{\sqrt{abc}}{\sqrt{a}-\sqrt{b}-\sqrt{c}} \\ \xi_2 &= \frac{\sqrt{abc}}{\sqrt{b}-\sqrt{c}-\sqrt{a}} \\ \xi_3 &= \frac{\sqrt{abc}}{\sqrt{c}-\sqrt{a}-\sqrt{b}} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (29)$$

These follow from the theorem by substituting for λ the values of the parameters of the vertices from equations (6)

Cor. 2. The radii of curvature at the vertices D, A, B, C are σ_4, σ_1, σ_2, σ_3, where

$$\left.\begin{aligned} \sigma_4 &= \frac{(\sqrt{b}+\sqrt{c})\,(\sqrt{c}+\sqrt{a})\,(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \\ \sigma_1 &= \frac{(\sqrt{b}+\sqrt{c})\,(\sqrt{c}-\sqrt{a})\,(\sqrt{b}-\sqrt{a})}{\sqrt{b}+\sqrt{c}-\sqrt{a}} \\ \sigma_2 &= \frac{(\sqrt{c}+\sqrt{a})\,(\sqrt{b}-\sqrt{a})\,(\sqrt{b}-\sqrt{c})}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \\ \sigma_3 &= \frac{(\sqrt{a}+\sqrt{b})\,(\sqrt{c}-\sqrt{a})\,(\sqrt{c}-\sqrt{b})}{\sqrt{a}+\sqrt{b}-\sqrt{c}}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (30)$$

These values are obtained by subtracting the abscissae of the centres of curvatures at the vertices, as given in the last Corollary, from the abscissae of the vertices, given in equations (6). Accordingly, when positive, they denote concavity to the right; and when negative, concavity to the left.

Cor. 3. The curve is concave to the right at the vertices D and A, and concave to the left at the vertex B.

Cor. 4. At the vertex C the curve is concave to the right, or to the left, according as

$$\sqrt{a}+\sqrt{b} > \text{or} < \sqrt{c}. \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

When at C the curve is concave to the right, we shall say the curve is an *indented* oval; if to the left, an *unindented* oval.

Cor. 5. The vertex C is a point of undulation, if

$$\sqrt{a}+\sqrt{b}=\sqrt{c}. \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Cor. 6. The radius of curvature at D is greater than the radius of curvature at A, and the latter greater than the radius of curvature at B, in absolute magnitude.

Cor. 7. In the unindented oval, the radius of curvature at C is greater that at D in absolute magnitude.

Cor. 8. On every indented oval there is a pair of real points of parameter zero.

For, the outer oval is traced as λ varies from

$$\lambda_4=\sqrt{(abc)}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \text{ to } \lambda_3=\sqrt{(abc)}(\sqrt{c}-\sqrt{a}-\sqrt{b})$$

and zero lies between these values.

Cor. 9. In every indented oval, there is an obvious double tangent. Its points of contact are the points of zero parameter.

For, the normal at either point cuts the axis at an infinite distance. Hence

$$\frac{abc}{\lambda}=\infty \quad \therefore \lambda=0.$$

Cor. 10. The distance of the points of contact of the double tangent from F_1, F_2, F_3 and O are

$$\sqrt{(bc)},\ \sqrt{(ca)},\ \sqrt{(ab)} \text{ and } \sqrt{(bc+ca+ab)}. \quad \ldots \quad (33)$$

For, making $\lambda=0$ in the fundamental parametric equations, the result follows.

Geometrical Investigation of the Secondary Systems of Bitangential Circles.

22. Let P, p_1 be a pair of inverse points on the curve, which are inverses in respect of the circle F_1. Let the normals at P, p_1 meet at g_1.

By the properties of inverse curves we have $g_1P = g_1p_1$. The circle of centre g_1 and radius g_1P or g_1p_1—which obviously touches the curve at P, p_1—will cut the circle F_1 orthogonally, as it passes through a pair of inverse points with respect to it, namely P and p_1.

To find the locus of g_1 as P varies: From O draw perpendiculars on the tangents to the oval at P, p_1 cutting the radius vector F_1p_1P at M_1 and m_1. By the property in question, we have $PM_1 = p_1m_1$, each being equal to $\sqrt{(bc)}$. Further they are in the same sense as will be obvious from the figure drawn with due regard to the observations in § 18. Consequently Pp_1 is equal to M_1m_1. Hence the triangles Pg_1p_1 and M_1Om_1, whose corresponding sides are parallel, are equal in all respects. Thus g_1P is equal and parallel to OM_1. Therefore Og_1 is equal and parallel to M_1P. Hence $Og_1 = \sqrt{(bc)}$ and the locus of g_1 is the 'generating' circle G_1. Therefore, if a variable circle move with centre on G_1, always cutting the circle F_1 orthogonally, its envelope is the Cartesian Oval. And a like proof holds for the *other* secondary systems of bitangential circles.

Cor. 1. Hence, P being any point on the oval, if parallels be drawn through O to the focal vectors PF_1, PF_2, PF_3 cutting the normal PG at g_1, g_2, g_3; then $g_1g_2g_3$ are the centres of the bitangential circles of the systems (G_1), (G_2), (G_3) which touch the curve at P.

Cor. 2. Consequently, the radii of the circles of the secondary systems of bitangential circles which touch the oval at P are equal to OM_1, OM_2 and OM_3.

Cor. 3. In the *indented* oval, the radii of the bitangential circles touching the curve at a point of contact of the double tangent are equal to OF_1, OF_2, OF_3; that is, are equal to a, b, c. ... (34)

Cor. 4. If g_1, g_2, g_3, G be the centres of the bitangential circles touching the oval at P, the set (P, g_1, g_2, g_3, G) is anharmonically constant.

For, it subtends at O a pencil, which, by parallels, is equal to that subtended at P, by a fixed set of points on the axis, namely

$$(O, F_1, F_2, F_3, \infty).$$

23. *The second property* of the triple focal perpendicular on the tangent consists of the following metrical relations.

Denoting OM_1, OM_2, OM_3 by n_1, n_2, n_3 we have

$$n_1 = \sqrt{\frac{a\rho_2\rho_3}{\rho_1}}, \quad n_2 = \sqrt{\frac{b\rho_3\rho_1}{\rho_2}}, \quad n_3 = \sqrt{\frac{c\rho_2\rho_1}{\rho_3}}. \qquad \ldots \quad (35)$$

Proof—Applying the equation of distances to connect OF_1, OM_1, OP and reducing, we get

$$\rho_1 . OM_1^2 = \frac{(\lambda + ab)(\lambda + ac)}{\sqrt{bc}} = a\rho_2\rho_3$$

Hence $n_1 = \sqrt{\frac{a\rho_2\rho_3}{\rho_1}}$, etc.

Cor. 1. From the last article it follows that *the values obtained are those of the radii of the secondary systems of* bitangential circles which touch the curve at the point λ.

Cor. 2. From the values of n_1, n_2, n_3 we get

$$\left.\begin{aligned} n_2 n_3 &= \sqrt{bc} \,.\, \rho_1 = \lambda + bc \\ n_3 n_1 &= \sqrt{ca} \,.\, \rho_2 = \lambda + ca \\ n_1 n_2 &= \sqrt{ab} \,.\, \rho_3 = \lambda + ab \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (36)$$

Cor. 3. Hence, in our figure,

$$OM_2 \,.\, OM_3 = PM_1, PF_1 ;$$

with two similar relations. (37)

Cor. 4. Also, by subtraction, we obtain

$$OM_1 . M_2M_3 = n_1(n_3 - n_2) = a(c - b) ;$$

and two similar relations. (38)

24. **Theorem :** If n denote the normal PG, then,

$$n = \frac{l_1}{\lambda}\sqrt{abc\rho_1\rho_2\rho_3}. \qquad \ldots \quad \ldots \quad (39)$$

For, by parallels,

$$PG : M_1O = PF_1 : M_1F_1 ;$$

that is

$$n : \sqrt{\frac{a\rho_2\rho_3}{\rho_1}} = \rho_1 : \frac{\lambda}{\sqrt{bc}}.$$

$$\therefore \qquad n = \frac{1}{\lambda} \sqrt{abc\rho_1 \rho_2 \rho_3}.$$

Cor. 1. We have

$$n_1 n_2 n_3 = n\lambda. \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

Cor. 2. $$n = \frac{1}{\lambda}\sqrt{(\lambda+bc)(\lambda+ca)(\lambda+ab)} \quad \ldots \quad \ldots \quad (41)$$

Cor. 3. If the abscissa of G be u, then,

$$n = \sqrt{\frac{(u+a)(u+b)(u+c)}{u}} \quad \ldots \quad \ldots \quad (42)$$

Cor. 4. Thus the radius of the circle, with centre at any point G on the axis and having double contact with the curve, is

$$\sqrt{\frac{GF_1 . GF_2 . GF_3}{GO}} \quad \ldots \quad \ldots \quad (43)$$

The circle is real provided G does not lie in the segments OF_1, F_2F_3.

25. By means of the last formula we can appreciate the effect produced on the oval when F_1, F_2. F_3 remain fixed, but the triple focus O moves to a considerable distance to the left. Let G be a typical point, first, in $F_1 F_2$; and secondly, in $F_1 F_3$ produced. The formula shows that when O recedes and GO becomes very large, the radius of the double contact circle, centre G, becomes, in each case, very small. Therefore, as regards the inner oval we infer that it tends to become very thin and look like a thin ellipse of foci F_1, F_2. As regards the outer oval, we realize the matter thus. As O moves, starting from a position near F_1, our oval is at first unindented. The indented form begins to appear as soon as O has passed a position O′ such that

$$\sqrt{O'F} + \sqrt{O'F_2} = \sqrt{O'F_3}.$$

As O recedes, the indentation, gentle at first, becomes very sharp, and very long, approaching the big cardioidal figure, with cusp at F_3, that the curve rapidly becomes. That this is the form can be gathered from the fact that when O recedes to a considerable distance the foci F_1, F_2, F_3 are coincident relatively to the distance of O from them. The double contact circles in question, with centres in $F_1 F_2$ produced are the circles inscribed in the creek of this cardioidal figure, whose form is like a thin parabola of focus F_3; and the circles inscribed in it become indefinitely small.

26. *To find the differential coefficients of* n_1, n_2, n_3 with respect to λ.

From the values $n_1 = \sqrt{\frac{a\ \rho_2 \rho_3}{\rho_1}}$, etc.,

and $$\frac{d\rho_1}{d\lambda}=\frac{1}{\sqrt{bc}},\quad \frac{d\rho_2}{d\lambda}=\frac{1}{\sqrt{ca}},\quad \frac{d\rho_3}{d\lambda}=\frac{1}{\sqrt{ab}},$$

we easily get

$$\left.\begin{aligned}\frac{dn_1}{d\lambda}&=\tfrac{1}{2}\cdot\frac{n_2+n_3-n_1}{n_2n_3}\\ \frac{dn_2}{d\lambda}&=\tfrac{1}{2}\cdot\frac{n_3+n_1-n_2}{n_3n_1}\\ \frac{dn_3}{d\lambda}&=\tfrac{1}{2}\cdot\frac{n_1+n_2-n_3}{n_1n_2}\end{aligned}\right\}\quad \ldots \qquad \ldots \qquad \ldots \quad (44)$$

We thus meet with a set of simultaneous differential equations, easily solved, to which the geometry of the cartesian oval affords an illustration.

27. It is interesting to see that when n_1, n_2, n_3 and n are given, we can find a, b, c and λ.

For, λ is at once given by

$$n_1n_2n_3=n\lambda.$$

When λ has been found, the values of bc, ca, ab are given by $n_2n_3=\lambda+bc$, etc.; whence a, b, c are known.

Hence, given the four bitangential circles of a cartesian oval touching the curve at a point P, the circle belonging to the axial system being distinctly specified, we can determine the oval.

Section V.

The Third Property of the Triple focal Perpendicular.

28. The property to be now considered is one of special importance. OQ being, as before, the triple focal perpendicular on the tangent at P, cutting the focal vectors PF_1, PF_2, PF_3 in M_1, M_2, M_3; the property in question is $OQ=\frac{1}{2}(OM_1+OM_2+OM_3)$... (45)

Denoting OQ by p, our Proposition is $p=\frac{1}{2}(n_1+n_2+n_3)$. ... (46)

Proof: In OQ produced take M_1' the reflexion of M_1 in the tangent at P.

Then, rect. $OM_1.OM'_1=OQ^2-M_1Q^2$;

$=OP^2-M_1P^2$;

$=(ab+ac+bc+2\lambda)-bc$,

$=(ab+\lambda)+(ac+\lambda)$;

$=n_1(n_2+n_3)$.

But $OM_1=n_1$

$\therefore$ $OM_1'=n_2+n_3$

$\therefore$ $OQ=\frac{1}{2}(OM_1+OM_1')=\frac{1}{2}(n_1+n_2+n_3)$.

That is, $p=\frac{1}{2}(n_1+n_2+n_3)$.

29. We shall now introduce three new quantities which are important in the analysis of the oval. These are the lengths M_1Q, M_2Q and M_3Q. We shall denote them by q_1, q_2, q_3.

We evidently have

$$p = n_1 + q_1 = n_2 + q_2 = n_3 + q_3 \quad \ldots \quad \ldots \quad (47)$$

From these and the result of last article, we get

$$p = q_1 + q_2 + q_3 \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

which is *another form* of the property in question. It shows that, in our figure,

$$QO = QM_1 + QM_2 + QM_3 \quad \ldots \quad \ldots \quad (49)$$

From (47) and (48), we get

$$n_1 = q_2 + q_3 ; \quad n_2 = q_3 + q_1 ; \quad n_3 = q_1 + q_2 ;$$

whence

$$\left.\begin{aligned} q_1 &= \tfrac{1}{2}(n_2 + n_3 - n_1) \\ q_2 &= \tfrac{1}{2}(n_3 + n_1 - n_2) \\ q_3 &= \tfrac{1}{2}(n_1 + n_2 - n_3). \end{aligned}\right\} \quad \ldots \quad \ldots \quad (50)$$

30. We have already denoted the length PQ by t.

In terms of t we obviously have

$$\left.\begin{aligned} q_1 &= \sqrt{bc - t^2} \\ q_2 &= \sqrt{ca - t^2} \\ q_3 &= \sqrt{ab - t^2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

Hence, substituting in (48), we get

$$p = \sqrt{bc - t^2} + \sqrt{ca - t^2} + \sqrt{ab - t^2} \quad \ldots \quad \ldots \quad (52)$$

which is the p, t equation of the Cartesian Oval.

The p, t equation is important in any curve as, *inter alia*, it gives a grip over the curvatures of the curve. And when, as here, we have p given *explicitly* as a function of t, the convenience is the greater.

31. Let us investigate the general p, t equation $p = \phi(t)$. ...(i)

Let ω be the angle which the perpendicular makes with a fixed line, this angle being so reckoned that we have

$$t = \frac{dp}{d\omega}.$$

From (i), differentiating with respect to ω, we get

$$t = \phi(t) \cdot \frac{dt}{d\omega};$$

Hence,

$$\frac{dt}{d\omega} = \frac{d^2p}{d\omega^2} = \frac{t}{\phi'(t)}. \quad \ldots \quad \ldots \quad (ii)$$

The radius of curvature being denoted by σ, we have

$$\sigma = p + \frac{d^2p}{d\omega^2}.$$

That is
$$\sigma = \phi(t) + \frac{t}{\phi'(t)} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Let σ_1 denote the radius of curvature of the evolute of the curve. Then

$$\sigma_1 = \frac{d\sigma}{d\omega}.$$

Now the operation

$$\frac{d}{d\omega} = \frac{dt}{d\omega}.\frac{d}{dt} = \frac{t}{\phi'(t)}.\frac{d}{dt}.$$

Applying these equivalent operations to σ, we get

$$\sigma_1 = \left\{ \frac{t}{\phi'(t)}.\frac{d}{dt} \right\} \left\{ \phi(t) + \frac{t}{\phi'(t)} \right\}. \ldots\ldots \quad \ldots \quad \text{(iv)}$$

And the radius of curvature of the n^{th} evolute is given by

$$\sigma_n = \left\{ \frac{t}{\phi'(t)''} \frac{d}{dt} \right\}^n \left\{ \phi(t) + \frac{t}{\phi'(t)} \right\}. \quad \ldots \quad \text{(v)}$$

32. Let us apply the formula obtained above, namely,

$$\sigma = \phi(t) + \frac{t}{\phi'(t)},$$

to the case of our oval.

We have $\phi(t) = \sqrt{bc - t^2} + \sqrt{ca - t^2} + \sqrt{ab - t^2}$; and we have the letters q_1, q_2, q_3 to denote the radicals involved.

Differentiating, $\phi'(t) = \frac{-t}{q_1} + \frac{-t}{q_2} + \frac{-t}{q_3}$.

$$\therefore \quad \frac{\phi'(t)}{t} = -\frac{q_1q_2 + q_2q_3 + q_3q_1}{q_1q_2q_3}.$$

Hence $\sigma = \phi(t) + \frac{t}{\phi'(t)} = (q_1 + q_2 + q_3) - \frac{q_1q_2q_3}{\Sigma(q_1q_2)}$.

$$\therefore \quad \sigma = \frac{(q_1+q_2)(q_2+q_3)(q_3+q_1)}{q_1q_2 + q_2q_3 + q_3q_1}. \quad \ldots \quad \ldots \quad (53)$$

For example, at the vertex D, $t = 0$ and we have

$$q_1 = \sqrt{bc},\ q_2 = \sqrt{ca},\ q_3 = \sqrt{ab};$$

and we get for the radius of curvature the value

$$\frac{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$$

which verifies the result already obtained.

33. Working out the general formula in § 31 for the radius of curvature of the evolute, we get

$$\sigma_1 = \frac{3tq_1q_2q_3(q_1+q_2)(q_2+q_3)(q_3+q_1)}{(q_1q_2+q_2q_3+q_3q_1)^3} \quad \ldots \quad \ldots \quad (54)$$

The evolute has cusps (1) at the centres of curvature at the vertices, for which $t=0$; and (2) at the centres of curvature at the points of contact of the tangents drawn to the oval from the external focus, at which points $q_3=0$. These are the only real cusps. The evolute is thus visibly, an 8 cusped curve. Four of these cusps relate to the inner oval, and four to the outer oval. In the unindented oval the evolute corresponding to each constituent oval has a look like that of the evolute of an ellipse. And the reader interested can readily find out the modification of form produced when the curve is an indented oval.

34. There is another method of investigating the radius of curvature at any point. At the very outset, in § 20, we met with the small inconspicuous equation

$$t = \frac{d\lambda}{ds}.$$

Now, $$t = \frac{dp}{d\omega}.$$

$$\therefore \quad \frac{dp}{d\omega} = \frac{d\lambda}{ds}.$$

Hence, we derive $$\frac{d\omega}{ds} = \frac{dp}{d\lambda}.$$

That is, $$\frac{1}{\sigma} = \frac{dp}{d\lambda}.$$

But $$p = \tfrac{1}{2}(n_1+n_2+n_3)$$

$$\therefore \quad \frac{1}{\sigma} = \tfrac{1}{2}\left\{\frac{dn_1}{d\lambda}+\frac{dn_2}{d\lambda}+\frac{dn_3}{d\lambda}\right\}.$$

And, we found in § 26 that

$$\frac{dn_1}{d\lambda} = \tfrac{1}{2}\,\frac{n_2+n_3-n_1}{n}\text{; etc.}$$

Hence we get the formula.

$$\frac{4}{\sigma} = \Sigma\,\frac{n_2+n_3-n_1}{n_2n_3}. \quad \ldots \quad \ldots \quad \ldots \quad (55)$$

This is readily transformed into that of § 32.

For $$q_1 = \tfrac{1}{2}(n_2+n_3-n_1),\ n_1 = q_2+q_3,\ \text{etc.}$$

Hence
$$\frac{4}{\sigma}=\Sigma\ \frac{2q_1}{(q_1+q_2)\ (q_1+q_2)}$$
$$=\Sigma\ \frac{2q_1(q_2+q_3)}{(q_1+q_2)\ (q_2+q_3)\ (q_3+q_1)}$$
$$=\frac{4\ \Sigma q_1q_2}{\Pi\ (q_1+q_2)}$$
$$\therefore\qquad \sigma=\frac{\Pi\ (q_2+q_3)}{\Sigma\ (q_1q_2)}.$$

35. The formula obtained for the radius of curvature admits of an important transformation which yields several simple geometrical constructions for the centre of curvature.

From the triangle OPQ, we have
$$OP^2=PQ^2+QO^2.$$
$$\therefore\qquad bc+ca+ab+2\lambda=t^2+(q_1+q_2+q_3)^2$$
$$=t^2+(bc-t^2)+(ca-t^2)+(ab-t^2)+2\ \Sigma\ q_1\ q_2.$$
Hence we obtain the important result
$$\lambda=-t^2+\Sigma q_1\ q_2, \qquad \ldots \qquad \ldots \qquad (56)$$
which expresses λ in terms of t_2.

This gives
$$\Sigma q_1\ q_2=\lambda+t^2.$$
Hence, from the formula
$$\sigma=\frac{(q_1+q_2)\ (q_2+q_3)\ (q_3+q_1)}{q_1q_2+q_2q_3+q_3q_1}$$
we get
$$\sigma=\frac{n_1n_2n_3}{\lambda+t^2}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (57)$$

36. Now we saw already that
$$n_1n_2n_3=n\ \lambda,$$
Hence we have
$$\frac{\sigma}{n}=\frac{\lambda}{\lambda+t^2}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (58)$$
This gives the following construction :

Let P be a point on the oval. Join PF_1 and cut off $PM_1=\sqrt{bc}$. Join OM_1 and produce it and draw PQ perpendicular to it. Then PQ is the tangent at P. Draw PG parallel to OQ cutting the axis in G. Then PG is the normal.

Now along PQ take Q_o, so that
$$PQ.\ QQ_o=PM_1\ .\ M_1F_1,$$
Also along PQ take P_o so that $PP_o=QQ_o$. Join Q_oG; and, through P_o draw P_o S parallel to Q_oG, cutting PG in S. Then S is the centre of curvature at P.

For, by parallels

$$PS : PG = PP_o : PQ_o ,$$
$$= QQ_o : PQ + QQ_o, \text{ since } PP_o = QQ_o$$
$$= PQ - QQ_o : PQ^2 + PQ \,.\, QQ_o.$$

Now $PQ.\ QQ_o = PM_1 \,.\, M_1F_1$, by construction $= \lambda$, by its geometrical meaning.

$$\therefore \qquad PS : PG = \lambda : t^2 + \lambda$$

i.e.,
$$PS : n = \lambda : t^2 + \lambda.$$

$$\therefore \qquad PS = \sigma, \qquad \text{by (58).}$$

37. Since $n_2 n_3 = \sqrt{bc} \,.\, \rho_1$, the relation (57) gives

$$\frac{\sigma}{n_1} = \frac{\sqrt{bc} \cdot \rho_1}{\lambda + t^2} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (59)$$

which leads to the following construction :

Draw the normal as before and along it, take $Pg_1 = OM_1$; from Q draw QX perpendicular on PM_1 ; along PM_1 take $XX_1 = M_1F_1$; join X_1 to g_1 ; through F_1 draw F_1S parallel to X_1g_1 cutting the normal at S. Then S is the centre of curvature required.

For, by parallels

$$PS : Pg_1 = PF_1 : PX_1$$
$$= PF_1 : PX + XX_1 ;$$
$$= PF_1 \,.\, PM_1 : PX \,.\, PM_1 + PM_1 \,.\, XX_1.$$

But
$$PF_1 \,.\, PM_1 = \rho_1 \,.\, \sqrt{bc}$$
$$PX \,.\, PM_1 = PQ^2 = t^2$$
$$PM_1 \,.\, XX_1 = PM_1 \,.\, M_1F_1 = \lambda.$$

$$\therefore \qquad PS : Pg_1 = \rho_1\sqrt{bc} : t^2 + \lambda$$

ie
$$\frac{PS}{n_1} = \frac{P_1\sqrt{bc}}{\lambda + t^2.}$$

Hence by (59) $PS = \sigma$, which proves the construction.

We have two *other* similar constructions.

"Some difficulties met with in reading Mathematics without a Teacher : A Complaint against Text-books."*

By. Major W. A. Garstin, Peshawar.

I was very pleased when the opportunity was offered to me of joining this Association at its start and I readily took advantage of the occasion because, although I cannot say that I am a mathematician, I do lay claim to being one of those others interested in mathematics to encourage and help whom is one of the primary objects of this Association. While, then, I welcomed the opportunity of becoming a member of the Association it never occurred to me that I should ever have the fortitude to read a paper before its members. And I do not doubt that I should never have done so had not an encouraging word been given to me by a friend,—himself a distinguished mathematician. We were discussing the methods and aims of mathematical teaching, and I was telling him of some of the difficulties which I had met with in reading the recognized text-books—difficulties which I was emboldened to maintain were due as much to defective methods of presentation as to my own dullness. The result of that brief discussion is the paper I am now to read to you. I cannot hope to bring to your notice anything very original ; indeed, it would seem presumptuous for anyone like myself, with no experience whatever as a teacher to address an audience such as this, on a subject connected with the vexed and difficult question of mathematical teaching. But I feel I shall not have wasted your time if I am able to bring home to those of you whose profession it is to teach, some of the inherent difficulties met with by one who has come to mathematics with the sole desire to learn and understand.

The title which I have chosen for the subject matter of this paper, "Some difficulties met with in reading Mathematics without the aid of a Teacher: A complaint against Text-books," suggests at once he comment that the difficulties must have been both various and many. That certainly is true; but the difficulties about which I am going to address you this afternoon I have narrowed down to a few of a more or less general character, such as are inherent in a subject as abstract as mathematics. They have nothing to do with the application of mathematics, and I am not going to trouble you about the hard points of this or that particular theorem. Very naturally I have encountered many difficulties of this latter kind where the assistance of a teacher would have been invaluable to me ; but my object in the present paper is altogether a different one. It is nothing more or less than to ask

* Communicated by the author to the Panjab Mathl. Association, March 1919.

you to consider some root matters of prime importance, in the presentation of which, so it seems to me, our English text-books fail lamentably to give that instruction which, in my opinion, it is legitimate to expect them to provide. They are difficulties relating to the fundamental ideas of mathematics as represented (or supposed to be represented) in our modern elementary text-books. Now it seems to me to be of the utmost importance that the mathematical student should acquire at, as early a stage in his mathematical reading as possible, a firm and clear grasp of the fundamental ideas of his subject, and should have a thorough understanding of its aim and object and the nature of the methods by which it works, so that he may gain a due appreciation of what mathematics as a science really is. This need one would have thought was so obvious that one could hardly imagine any text-book professing to teach mathematics not entering upon at least some discussion of these matters; yet it has been my experience that it is just in regard to them that the elementary text-books are profoundly silent; either because, I suppose, the ideas are considered not to be suitable for the beginner or because they are not necessary for the purpose of mathematics, as that hard-grained subject is known in the schools. To neither of these views do I subscribe; and in my opinion it is because this side of the subject is so neglected, that not only is mathematics looked upon by the ordinary student as a mere mechanical juggling with symbols, but students with marked mathematical ability shew themselves to possess but the most confused and hazy ideas of its fundamental concepts. I do not mean, I may hasten to explain, anything of a metaphysical nature, such as how it is we cannot conceive that two and two make five (though some of us have been known, I daresay, to arrive at that result in our everyday task)—that part of the *theory* of mathematics belongs to the philosophy of the subject, of which Mr. Bertrand Russel is so able and well-known an exponent. But there are ideas fundamental to mathematics which are, I think, quite inadequately treated in our elementary text-books, whereas too much time, and space to consider, cannot be spent in making such notions clear I venture and in taking care that by means of numerous examples and exercises the student really does understand them.

An excellent instance of the kind of difficulty I am dealing with is given by Mr. Hardy in the preface to the first edition of his "*Pure Mathematics*". He is writing about the notion of a limit, a notion "that", he writes, "has always presented grave difficulties to mathematical students even of great ability". It has been his good fortune, he explains, to have had a share in the teaching of a good many of the

ablest candidates for the Mathematical Tripos and, to quote his own words: "it is very rarely indeed that I have encountered a pupil who could face the simplest problem involving the ideas of infinity, limit or continuity with a vestige of the confidence with which he would deal with questions of a different character and of far greater intrinsic difficulty." "I have indeed", he goes on to say, "in an examination asked a dozen candidates, including several future Senior Wranglers to sum the series $(1+x+x^2+\ldots\ldots)$ and not received a single answer that was not practically worthless——and this from men quite capable of solving difficult problems connected with the curvature and torsion of twisted curves". "I cannot believe", he says, "that this is due solely to the nature of the subject," and, then, after admitting that, no doubt, there are difficulties in those ideas but that they are not so great as many other difficulties inherent in mathematics which every young mathematician completely overcomes, Mr. Hardy arrives at the conclusion that "the fault is not that of the subject or of the student, but of the text-book and the teacher". These words by an eminent authority exactly represent the kind of difficulty which I have encountered in reading by myself—difficulty in getting at the bed-rock of the matter and not at attaining mere technical skill. For, I take my stand on "mathematics for mathematics' sake", and not for the sake of passing an examination or necessarity applying it. And the illustrations upon which I will now enter are therefore those which have chiefly to do with pure mathematics.

The first point to which I would draw your attention is the lack in the current elementary text-books of any adequate explanation of what is the aim and object of (pure) mathematics and how its various branches have arisen; no attempt is made to view the subject as a whole so that the great underlying principles which govern the subject may be seen and the student thereby led to realize that all its various activities, algebra, trigonometry, geometry, analyticalgeometry and their multifarious developments, are indeed branches of a parent stem, are, what Browning would call, 'the broken arcs of a perfect whole.' What is it then that the Mathematician is striving to obtain? He starts with mere numbers and a relation like $2+3=5$; but he soon begins to generalize and replaces these particular figures by the general letters a, b, and x; that is, he replaces "some" by "any", and is thus led on to a world of general theorems and deductions. But the content of his letters begin to extend by the conception of positive and negative quantity, and the representation of these two different concepts gives rise to an entirely new branch of mathematics: namely, analytical

geometry. Then again the treatment of equations of the second degree only led to solutions which for many ages were regarded simply as 'impossible', but to interpret which became imperatively neceessary if the science was not to be impeded at every fresh advance by ever-increasing restrictions and many interesting results were not to remain meaningless. Thus new symbols were invented and old ones made to clothe ideas vastly extended beyond that of mere number or quantity, but yet so defined as to obey the fundamental laws of algebra. And finally came the epoch-making recognition of the two different aspects of quantity as discrete and as continuous, with the resultant analysis reared on the basis of continuity. In all this, what the mathematician is really seeking is 'Generality,' a progress from the idea of 'some' things to 'any' things, an advance from the particular to the general. And with this idea of Generality go the ideas of the Variable and of Form to form, what Dr. Whitehead has aptly called, the Mathematical Trinity. This trinity forms the parent stem from which all branches spring and have their being. Now where in any current text-book has any full and clear exposition been given of these fundamental notions at all comparable to that given by Dr. Whitehead in his "*Introduction to Mathematics*"—a book costing one shilling and easily understood by any boy? Certainly not in any text-book I have ever met. Indeed, what I believe are commonly used as text-books make no attempt at all at any such explanation; and yet the ideas are fundamental to mathematics and must be thoroughly grasped if any proper understanding is to be gained of the object and aim of mathematics. My experience has been that the various divisions of the subject are treated as so many water-tight compartments, so many isolated chapters having no connecting thread between them, with the result that the subject of, say, imaginary qnantities looks like a mere excursion into the fantastic.

The mention of imaginaries furnishes me with a good illustration to point. The introduction of the student to complex numbers which our elementary text-books provide is, to my mind extraordinarily meagre and unsatisfactory. While one book may give the briefest possible explanation of the symbols used, another is content with a bald definition of a complex number and its representation graphically, a third may be a little fuller but still utterly fail to convey a proper understanding of the part played by the use of these symbols in mathematical analysis in order that the student may grasp the basic idea underlying their conception; how their introduction has given complete *generality* and *symmetry* (the notion of form again) to the

theory of algebraic equations. I wonder whether one of Mr. Hardy's clever students would have had as much difficulty in giving a plain and convincing explanation of what the symbol $(a+bi)$ means, how it arose, and the reason for its choice, as he had in giving an intelligible explanation of the meaning of the 'sum to infinity' of a geometric series? The important fundamental ideas are the symbolical nature of these so-called imaginary expressions, the manner in which a concrete representation is given to this symbolism by employing the usual system of rectangular axes in a plane, the complex quantity $(a+bi)$ being represented by the point M in the plane XOY whose co-ordiantes are $x=a$ and $y=b$, and the fact that to every proposition established for complex quantities there is a corresponding theorem of geometry, though these are not the greatest advantages arising from their representation. Real numbers correspond to the x—axis, which for this reason is called the *axis of reals*. If the student is to get clear ideas of the meaning and use of these symbolical expressions then it seems to me that a great deal more explanation and illustration is needed than is usually given in the ordinary text-book. After all the ideas are not easy, and unless the nature of the symbols used is made clear, the integral part they play in mathematical analysis explained, and the mode in which they arose traced, the reader of the text-book, although he may be able to solve many academic exercises on the application of DeMoivre's theorem, will have but the haziest idea of what is really important, namely, the splendid mathematical conceptions which made all these remarkable results possible. In explaining the most elementary ideas of all connected with limits, Mr. Hardy devotes a chapter of fifty pages without advancing beyond the ordinary geometric series; and yet I am sure that the student who has digested those pages will have gained much more lasting (mathematical) profit than one who has solved all the examples on geometrical progression in Hall & Knight's "*Higher Algebra.*" Again take the notion of a *function*. Compare, for instance, Mr. Hardy's treatment of classes of functions in chapter II of his book with what is considered sufficient in a text-book, say, of the class of C. Smith's *Algebra*. As Mr. Hardy himself points out in the preface of his book, his treatment is "more systematic and illustrated with much greater detail than is usual in English books." He is at pains to bring out, and he succeeds admirably in doing so, the really important point in functional relationship, $y=f(x)$, and that is that any absolutely arbitrary law may be assumed for finding the value of y from that of x. The word *function* in its most general sense means nothing more or less than this; to every value of x corresponds a value of y. This is the important part of the matter; let it be driven home, as Mr.

Hardy drives it home, by a wealth of illustrations and exercises. And so in Edouard Goursat's (Professor of Mathematics in the University of Paris) preface to his *Mathematical Analysis.*, I read "Since mathematical analysis is essentially the science of the continuum, it would seem that every course in analysis should begin, logically, with the study of irrational numbers. The thoery of incommensurable numbers is treated in so many excellent well-known works that I have thought it useless to enter upon such a discussion." Well, as the authors of the American translation of the work remark, such "excellent well-known books" are *not* common in English.

I would again remind you that the difficulties to which I refer in this paper are not those connected with the application of mathematics but arise out of what I consider to be the inadequate representation of its fundamental ideas. My next example, however, is of a more specific kind in that it deals with a particular theorem, but it illustrates well the common failure of the ordinary text-book to explain the precise nature of the conception on which the mathematical theory is based. To the student, when first introduced to it, the mathematical theory of probability appears as some thing singularly ineffectual, either because it is so against his common experience, as when for instance he does happen to play in a game of chance, the success or failure of his particular venture seems to upset entirely the mathematical theory, or because he is asked to consider such examples as the throwing of ideally perfect dice or the tossing of ideally perfect coins—things to be classed with such mathematical fictions as frictionless pulleys and weightless elephants! Now, of course, these perfect conditions are only chosen to fix the mathematical ideas; and the conditions are necessarily conceived as ideal because the mathematical theory of probability deals with the ideal since it aims at results which are to be exact. It must be made clear to the student that 'probability' as a mathematical quantity is not, as Prof. Chrystal well expresses it, an attribute of any particular event happening on any particular occasion, but can only be predicated of an event happening or conceived to happen, "on the average" or "in the long run." The important idea here then is that of happening a great many times; and I agree with Prof. Chrystal that the word *frequency* would be a much better word to use than *probability* to convey the idea of a very large number of occasions. These considerations are not, I think, sufficiently insisted upon in the proofs of the elementary theorems of the subject. I take as my example the proof, as I usually find it given in the text-books, that the probability of an event being known, it is required to find the probability of its happening exactly *r*

times in n trials. The usual text-book treatment is that if any particular r sets are selected out of the total number n the chance that the event will happen in every one of these r trials and fail in all the rest is $p^r q^{n-r}$, by the law of independent events; from which it follows that the terms in the expansion by the binomial theorem of $(p+q)^n$ represent respectively the required probabilities. Now, compare the above treatment with the following which I have taken from an elementary text-book on the theory of statistics:—"If we deal with one event only, we expect in N trials, Nq failures and Np successes. Suppose we combine with the results of this first event the results of a second. The two events are quite independent, and therefore, according to the rule of independence, of the Nq failures of the first event $Nq.\ q$ will be associated, *on an average*, with failures of the second event, and $Nq.\ p$ with successes of the second event. Similarly of the Np successful first events $Np.\ q$ will be associated, on an average, with failures of the second event and $Np.\ p$ with successes. In trials of two events we would therefore expect approximately Nq^2 cases of no successes, $2Npq$ cases of one success and one failure, and Np^2 cases of two successes. The results of a third event may be combined with those of the first two in precisely the same way, to expect, on the average, Nq^3 cases of no successes, $3\ Npq^2$ cases of one success, $3\ Np^2q$ cases of two successes, and Np^3 cases of three successes. Similarly for a fourth event. Hence it is evident that all the results are included under a very simple rule: the frequencies of 0, 1, 2, ...successes are given

for one event	by	the	binomial	expansion $N\ (p+q)$
for two events	„	„	„	$N\ (p+q)^2$
for three events	„	„	„	$N\ (p+q)^3$
for four events	„	„	„	$N\ (p+q)^4$

and so on. Quite generally in fact: the frequencies of 0, 1, 2... success in N trials are given by the successive terms in the binomial expansion of $N(p+q)$."

The above method seems to me to supply just the necessary connexion between theory and fact to let the student see how the mathematical theory is the ideal.

I should like to give you more examples of the same nature in which the main object of the text-books appears to be to get the student through a number of theorems as quickly as possible to the sole end that he may gain facility in mere technique. The treatment of series would furnish me with an excellent instance in point—the binomial

expansion, the exponential series, the logarithmic series, and the various trigonometrical series. They are treated without any attempt to explain —and such explanation I submit should,come very early—how they all lead up to and form part of the general problem of the development of a function as a power series and that to be able so to develop a function is a great advantage. For example, the most effective way of defining $\sin x$ and $\cos x$ is not by the usual geometrical method bnt as limits respectively of the series.

$$\sin = x - x^3/3\,! + x^5/5\,! - \ldots$$
$$\cos x = 1 - x^2/2\,! + x^4/4\,! - \ldots$$

Indeed unless a student proceeds as far as the Calculus, and there are many students for whom mathematics forms a part of their curriculum, who never get as far as that, he never is told that there are such theorems as Taylor's and Maclaurin's; he never learns that all those various series are but particular cases of a general theorem which for the width and the general character of its results is a marvellous achievement. But I must not detain you longer by multiplying illustrations as, before closing, I wish to say a few words about a subject on which a great mass of literature has accumulated, and I hope that the instances I have cited will have sufficed to make my meaning clear to you.

I refer to the Differential Calculus; a big subject, but there is only one point in it to which I wish here to draw your attention. That is, the danger which the student who really tries to think things out might be led into of inferring, that the calculus is a method of approximation only; a very close approximation but still an approximation, a mere asymptotic approach to a value which is never actually reached. Thus in Lamb's excellent text-book, a numerical example of the manner in which the ratio $\delta y/\delta x$ approximates to its limiting value is given in the case of $y = \log_{10} x$ for the neighbourhood $x = 1$, $D_x y = .43429$; a table is worked out showing how for different values of δx in the neighbourhood of 1 and the corresponding values of δy, the ratio $\delta y : \delta x$ approximates closer and closer to the value of $D_x y$. Well the student might reasonably reply: Yes, I quite see that, but how does one pass from the state of getting nearer and nearer to the value to the state of actually reaching it? What he has read about limits preparatory to the calculus and what he reads when he comes to the calculus itself is so surrounded with such expressions as "in the neighbourhood of", "as we approximate to", "as we approach" and so on, that I think

it is not surprising if he asks: But when are we going to get there? His difficulty, of course, is the old one of the paradox of Achilles and the tortoise.—Achilles who never overtook the toroise although he got "as close as we please" (as the stock text-book phrase is) to doing so because while he was making up on the tortoise the latter travelled a certain fraction of the distance which Achilles had to cover to reach the spot where the tortoise was originally, and hence the tortoise still remained just that finite fraction ahead of him. The paradox is due to confusing two distinct kinds of motion, one discrete in which the passage is from one value to another by definite and separate stops or jumps, and the other continuous in which the change of position passes through all intermediate values. Zeno's famous paradox indeed is an excellent one to bring home to the student the difference between continuous and discontinuous variation. Now I think that the reason, or at least one reason, for the confusion of thought which might lead a student to imagine that the calculus is only an approximate method arises from making the idea of the limit subordinate to the calculus. It is, of course, nothing of the sort. As far as the doctrine of limits is concerned *qua* such, the calculus is a mere incident. I mean that in the notion of the limit,it is of capital importance for the student to understand thoroughly that the limit of a function is *not* its value. The limits may possibly be equal to the values of the function; but, as Mr. Hardy points out, whether it is so or not "has nothing whatever to do with the notion of the limit." Let the student grasp this fact and he will then realize what a very incidental affair the calculus really is—applying only to those common or garden functions which have a value (or values) equal to the limit, that is, those only which are continuous. Let the notion of a limit be drilled into the student before he passes to a study of the calculus and he will see that, as regards the calculus, the crux of the whole matter lies in the fact that the limit of $f(x+h)$, as h approaches zero, is $f(x)$, and that unless this is so, the differential coefficient cannot exist. At present, the limit is taught as something existing for the calculus with the consequence that there is a danger of the student never getting really to understand the notion of a limit and incidentally of gaining a false perspective of the true place of the calculus in the hierarchy of mathematical ideas.

I have been led to invite your attention to this point regarding the differential calculus because I see from the agenda of to-day's meeting a proposal that this Association should undertake to write a standard book on the Calculus to meet the requirements of the Punjab University students. This, if you will permit me to say so, savours

rather of a book to meet the requirements of the Punjab University examinations. May I express the pious hope that the examiners will combine with the authors and will rise to the occasion to produce a work which will merit the title of 'standard' in the best sense? that the authors will take a broad 'humane' view of their subject and will give us a treatise worthy the student of any University who has a real desire for knowledge, and not a work designed merely to give a facility in working mechanical exercises, which, however ingenious they may be, are of no educative value whatsoever. I deny that to teach a student that the differential coefficient of x^n is nx^{n-1} is to teach him the differential calculus. I quite see the other side of the question: thousands of persons can use the slide rule or a vernier without being acquainted with the theory of their construction. But I think the object of this Association is to foster a spirit less utilitarian than that. Approach the subject from the standpoint of "mathematics for mathematics' sake" and is it too much to hope that mathematics can be made to take its place among the humanities? Or, should we not at the most expect more than that such a 'value' should be the 'limit' to which we can approach as close as we please but never reach? Let it then be our ideal. It is as I have read mathematics by myself with this ideal before me that I have encountered those difficulties about which it has been my privilege to address you this afternoon.

Since writing the above my attention has been drawn to a new set of text-books on algebra (including trigonometry) by T. Percy Nunn in Longman's Modern Mathematical Series. I have had time only to give them a hasty perusal, but from what I have seen, they seem to me to be a thorough break-away from the older type of text-book against which my remarks have been directed. They appear to be a departure on the very lines I have been advocating, and I would earnestly commend them to your notice.

Astronomical Notes.

1. *The Problem of Cepheid Variation.*—In the Notes for June, mention was made, of the investigations by Prof. Eddington on the Dynamics of the Pulsation theory of the cepheid variables. Prof. Jean is a great opponent of this theory. His investigations on the subject have led him to a different conclusion. His discussion is based on a study of the light curves and spectral variations of these stars. He suggests that the physical cause of variation of the cepheids and long-period variables is the same, being of the nature of explosions or tidal eruptions on a rotating body, and that the explosions are of the nature of 'forced' and not 'free' oscillations.

2. *The parallax of the Pleiades.*—This group of brilliant stars has attracted attention from the earliest times. Several attempts have been made to determine the distance of this important star cluster. Prof. Kapteyn has developed an indirect method of finding the mean parallax of a cluster by counting the number of stars it contains under different magnitudes, assuming that the luminosity curve is known. Dr. Schonten of Holland has recently applied this method to ascertain the distances of some well-known clusters. The results in the case of thirteen of them are given in the Observatory No. 537. The parallax he obtains for the Pleiades group is 0″036 with a probable error of ± 0″010, which gives the distance of the cluster to be 88 light years. This is of the same order of magnitude as the parallaxes found by Kapteyn and Plummer by other methods.

3. In Astr. Nach 4994, an investigation has been made of the orbit of the Comet 1832 II, applying Einstein's Relativity Theory of Gravitation. The elements of this orbit are slightly different from those obtained from Newton's Laws. The period determined is 760.9 years, the Einstein effect being to shorten it by 10.9 years.

NIZAMIAH OBSERVATORY,
Hyderabad.

T. P. BHASKARA SASTRI.

SOLUTIONS.

Questions 991 and 992.

991. (S. NARAYANAN) :—ABC is a triangle whose incentre is I and orthocentre H. If the in-centre of the exmedial triangle of ABC be K and KH bisected at L prove that the nine-points centre of ABC is the middle point of LI.

992. (S. NARAYANAN) :—The internal bisectors of the angles A, B, C of a triangle meet the sides of the medial triangle DEF at P, Q, R. Prove that the conic which touches these sides at P, Q, R touches also the principal Feuerbach tangent of DEF.

Solutions by N. Sankara Iyer, L. N. Subramanyam, S. R. Ranganatham and M. K. Kewalramani.

If K is (α, β, γ) referred to $\triangle$ ABC, we get

$$\frac{2\triangle}{a}-\alpha=\frac{2\triangle}{b}-\beta=\frac{2\triangle}{c}-\gamma=2r.$$

We have thus to prove that

$$2\,(O_1+H_1)=2\,I_1+2L_1=2\,I_1+K_1+H_1,$$

where O_1, H_1, &c., denote the α—co-ordinates of the points ; *i.e*, we have to prove that

$$2\,O_1+H_1=2\,I_1+K_1,$$

i.e, $$2\,R\cos A+2\,R\cos B\cos C=2r+\frac{2\triangle}{\alpha}-2r=\frac{2\triangle}{\alpha},$$

i. e., $$\cos A+\cos B\cos C=\sin B\sin C$$

which is true.

For Question 992, we take D E F as the triangle of reference. It is clear that the internal bisector of A meets E F at the isotomic conjugate of the meeting point of E F and the internal bisector of D. The equation of this bisector is $\gamma=\beta$ or $\frac{z}{c}=\frac{y}{b}$ in areals. The equation of DP is therefore $\frac{1}{cz}=\frac{1}{by}$ in areals, or $b^2\beta=c^2\gamma$.

The required conic is therefore $\Sigma\sqrt{(a^2\alpha)}=0$.

The Feuerbach tangent is $\sum\frac{a\alpha}{b-c}=0$.

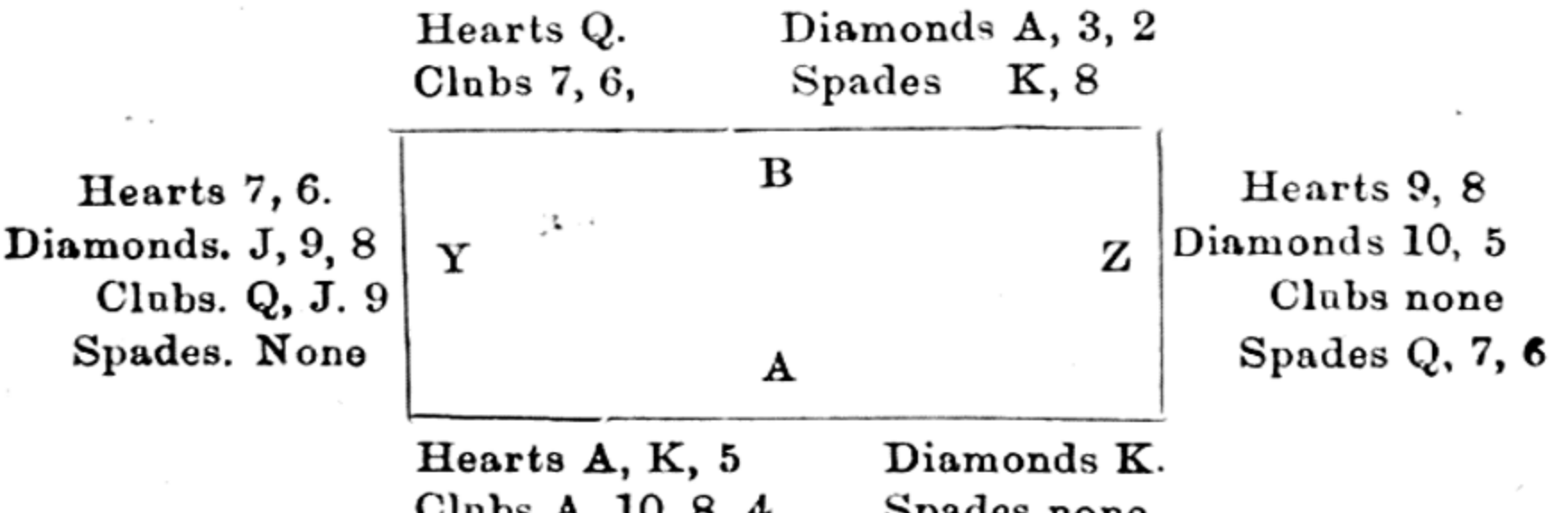
This will touch the ellipse, if

$$\sum \frac{a^2(b-c)}{a} = 0, \text{ i.e., } \Sigma a(b-c) = 0,$$

which is true.

Question 994.

(K. B. MADHAVA) :—Conduct the following game of Bridge with Hearts as trumps, in which A has to lead and A and B together have to make seven tricks.



Solution by R. Littlehailes, R. D. Karve, N. Sankara Aiyar, P. A. Subramani Aiyar E. Appalachari, and T. R. Narayana Aiyar.

(1)	A leads Heart A :	Y follows Heart 6 :	B, Heart Q :	Z Heart 8
(2)	A, Diamond K :	Y, Diamond 8 :	B, Diamond A ;	Z, Diamond 5.
(3)	B, Diamond 3 :	Z, Diamond 10 :	A, Trump K ;	Y, Diamond 9.
(4)	A, Heart 5 :	Y, Heart 7 :	B, Club 6 ;	Z Heart 9.
(5)	Z, Spade 7 :	A, Club 4 :	Y, Diamond J ;	B, Spade 8.
(6)	B, Spade K :	Z, Spade 5 :	A Club 8 ;	Y Club 9.
(7)	B, Diamond 2 ;	Z, Spade 6 :	A, Club 10 ;	Y Club J.
(8)	B Club 7 ;	Z, Spade Q :	A, Club A :	Y Club Q.

N.B.—Trick (4) is won by Z and all the others by A and B. The success of the game consists in playing into Z's hands at a time when he can play only spades thus giving B two tricks.

In tricks 5 and 6 it is open to Y to reserve Diamond J and throw down Club 9 and Club J ; in which case B leads Club 7 in trick (7) and A secures both the tricks 7 and 8.

QUESTIONS FOR SOLUTION.

1054. (MARTYN M. THOMAS) :—If s_r denote the sum of the squares of the fractions $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \ldots\ldots \frac{1}{r}$, and p_r denote the sum of their products taken two at a time, show that

$$nC_1\, s_1 - nC_2(s_2+p_2) + nC_3(s_3+p_3) - nC_4(s_4+p_4) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (-1)^{r-1} nC_r(s_r+p_r) = \frac{1}{n^2}.$$

1055. (ENQUIRER) :—Show that

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} = \frac{1}{2^2} + \frac{5}{2^7} + \frac{7\cdot 9}{2!}\frac{1}{2^{12}} + \frac{9\cdot 11\cdot 13\cdot}{3!}\cdot\ \frac{1}{2^{17}} \ldots\ldots\ldots$$

1056. (M. K. KEWALRAMANI) :—If

$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, show that

$$\sum \frac{\cos^2(p\alpha+q\beta+r\gamma)}{(2p)!\,(2q)!\,(2r)!} = \frac{\cos 2n\alpha + \cos 2n\beta + \cos 2n\gamma}{(2n)!}\cdot 2^{2n-2}$$

with a corresponding expression for sines; summation being extended to all positive integral values of p, q, r including zero, subject to the condition $p+q+r=n$.

1057. (M. K. KEWALRAMANI) :—Show that

$$\sum \frac{(a+b+c)^{2p}(b+c+d)^{2q}(c+d+a)^{2r}(d+a+b)^{2s}}{(2p)!\,(2q)!\,(2r)!\,(2s)!}$$

$$= \frac{2^{n-4}}{(2n)!}\left\{2\cdot 3^{-n}\sigma^{2n} + 2\Sigma^2(\sigma+a)^{2n} + \Sigma(\sigma-a-b)^{2n}\right\}$$

where $2\sigma = a+b+c+d$ and the summation extend to all positive integral values of p, q, r, s including zero but subject to the condition $p+q+r+s=n$.

1058. (N. P. PANDYA) :— A fixed ellipse is intersected by a variable parabola in four points A, B, C, D. If the axis of the parabola be always perpendicular to the axis major of the ellipse, find the locus of the centre of gravity of the quadrilateral ABCD.

1059. (K. J. SANJANA, M. A.) :—Prove that the sums of the products of the quantities $(2^2-1)^{-1}$, $(4^2-1)^{-1}$,, $(6^2-1)^{-1}$,...*ad inf.*, taken three at a time and four at a time, are respectively

$$\frac{10-\pi^2}{32} \text{ and } \frac{1680-180\pi^2+\pi^4}{6144};$$

and show how to find the sums of the products taken five, six,.....at a time.

1060. (K. J. SANJANA, M. A.) :—Show how to find the sum of the products r at a time of the quantities

$$(3^2-1)^{-1},\ (5^2-1)^{-1},\ (7^2-1)^{-1},\ldots\ldots ad\ inf.$$; and find the values for $r=3$ and $r=4$.

1061. (LAKSHMISHANKAR, N. BHATT) :—The Euler line of the triangle AB_1C_1 meets the sides AB_1, AC_1 in B_2, C_2 ; the Euler line of AB_2C_2 meets these sides in B_3, C_3 ; and this process is continued indefinitely. If N_r is the nine-point centre of the rth triangle AB_rC_r thus formed, and AN_r meets B_rC_r in D_r, prove that the straight line $N_x N_{x+2y}$ and $D_x D_{x+2y}$ are parallel, x and y being any positive integers.

1062. (C. KRISHNAMACHARI) :—Prove that

$$1+\binom{n-1}{1}\frac{1}{3}+\binom{n-1}{2}\frac{1}{5}+\ldots+\frac{1}{2n-1}$$

$$=2^{n-1}\left\{\frac{1}{2n-1}+\frac{n-1}{(2n-1)\ (2n-3)}+\right.$$

$$\left.\frac{(n-1)\ (n-2)}{(2n-1)\ (2n-3)\ (2n-5)}+\ldots+\frac{\lfloor n-1}{(2n-1)\cdot(2n-3)\ldots 1}\right\}$$

1063. (C. KRISHNAMACHARI) :—Show that

$$(a)\quad \frac{1}{2^{n+1}}\left\{\frac{1}{n}+\frac{2n-1}{n(n-1)}+\frac{(2n-1)\ (2n-3)}{n(n-1)(n-2)}+\ldots\ldots\right.$$

$$\left.+\frac{(2n-1)\ (2n-3)\ldots 3}{\lfloor n}\ \frac{\pi}{4}\right\}$$

$$=1-\ (n+1)\tfrac{1}{3}+\ \frac{(n+1)(n+2)}{2!}\ \tfrac{1}{5}-\ldots\ldots\ldots$$

$$(b)\quad 1-\ \binom{n-\frac{1}{2}}{1}\frac{1}{3}+\ \binom{n-\frac{1}{2}}{2}\frac{1}{5}-\ldots\ldots\ldots$$

$$=2^{n+\frac{1}{2}}\left[\frac{1}{2n+1}-\ \tfrac{1}{2}\binom{n-\frac{1}{2}}{1}\frac{1}{2n+3}+\frac{1}{2^2}\binom{n-\frac{1}{2}}{2}\frac{1}{2n+5}-\ldots\right]$$

1064. (S. KRISHNASWAMI IYENGAR) :—If in Question 986 we have a rectangular hyperbola instead of a parabola, prove that Q K

$$=\frac{(ds)^5}{120\rho^4}\left\{\rho^2\frac{d^3\rho}{ds^3}-\frac{1}{3}\rho\frac{d^2\rho}{ds^2}-6\rho\frac{d\rho}{ds}\frac{d^2\rho}{ds^2}+4\left(\frac{d\rho}{ds}\right)^3+36\frac{d\rho}{ds}\right\}.$$

1065. (S. KRISHNASWAMIENGAR):—If ρ be the radius of curvature of the curve $r^m=a^m \sin m\theta$ at the point whose distance measured along the curve from a fixed point is s, prove that

$$(m-1)(m+1)^2\rho\frac{d^2\rho}{ds^2}-m(m+1)^2\left(\frac{d\rho}{ds}\right)^2-m(m-1)^2=0.$$

Show how to solve this equation and hence to find the equation of the given curve in terms of ρ and s.

Similar formulae are got, by giving particular values to m, for a parabola, a rectangular hyperbola, a lemniscate and a cardiode.

1066. (P. A. Subramanya Iyer) :—Solve completely

$$a\,xy\left(\frac{dy}{dx}\right)^2+(x^2-ay^2-b)\,\frac{dy}{dx}-xy=0.$$

1067. (V. Tiruvenkatachari) :—Show that

$$\int_0^{\frac{1}{2}\pi}\frac{\log\sin x\log\cos x}{\sqrt{(\sin x\cos x)}}\,dx=\tfrac{1}{8}\,\frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}\left\{(\log 2)^2+\pi\log 2-\frac{\pi^2}{4}\right\}.$$

1968. (A. C. L. Wilkinson) :—E is any straight line parallel to the Euler line of a triangle ABC, E_1. E_2. E_3 are the Euler lines of the triangles formed by AB, AC, E ; BC, BA, E ; CA, CB, E ; prove that

(1) E_1. E_2. E_3, are parallel respectively to BC, CA, AB ;

(2) $E_1E_2E_3$ form a triangle equal in all respects to ABC ;

(3) the loci of the vertices of the triangles formed by $E_1E_2E_3$ as E varies are three straight lines parallel to the axis of the parabola which touches the sides of the triangle ABC and its Euler line ;

(4) if E passes through C, then E_1, E_2 intersect at the point where the Euler line of the triangle ABC meets AB ;

(5) the locus of the centre of homology of the triangles ABC, E_1 E_2 E_3 is a straight line parallel to the axis of the parabola defined in (3).

1069. (A. C. L. Wilkinson).—Normals PA, PB, PC, PD are drawn to a hyperbola ; PA meets the hyperbola again in A′, PB in B′, PC in C′, PD in D′. I(A′B′C′D′ are concyclic, prove that the locus of P is an ellipse of which the equi-conjugate diameters coincide with the asymptotes of the hyperbola.

1070. (S. Ramanujam, F.R.S.) :—Show that

(i) $$\sqrt{\sqrt[3]{\tfrac{1}{2}}+\sqrt[6]{\tfrac{4}{5}}}=\sqrt[6]{1+\sqrt[5]{2+\sqrt[6]{8}}}$$
$$=\sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}-\sqrt[5]{\frac{1}{125}}$$

(ii) $$\sqrt[3]{\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}}=\sqrt[5]{\frac{1}{25}}+\sqrt[5]{\frac{3}{25}}-\sqrt[5]{\frac{9}{25}},$$

(iii) $$\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}}=\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}.$$

Mathematics (in brief) from Current Periodicals.

The Mathematical Gazette, Vol. IX, No. 139, March 1919.

The Graphical Treatment of Differential Equations, by S. Brodetsky, Ph. D. (to be reported in a later number). *Mathematics and the Pivotal Industries*, by W. P. Milne. (The object of the paper is to discuss the the importance of the Mathematical Association lending a helping hand in drawing up suitable mathematical syllabuses for the various 'pivotal' industries such as, Agricalture, Commerce, Engineering, Mining &c., and to see whether the Mathematical Association itself, and indirectly the Secondary Schools, would not benefit by occasionally viewing mathematics from the different view-points of industry rather than by focussing the whole attention on the academic aspect of the subject). *The Teaching of Geometry to First Year Pupils* by B. A. Howard. (The stand point of the paper is that our chief aim should be the development of the reasoning faculty ; insisting always that this can only be done by sweeping away the abuses of Euclidean tradition and giving a more human and less philosophic orientation to our teaching. A schedule of first year work is outlined and discussed in the light of this statement of aim) *Cubic Graphs of the Form* $y=ax^3+bx^2+cx+d$, by A. Lodge. (The object of the paper is to call attention to some simple means of interpolating points in a cubic graph and of indicating gradients without actual calculation).

The Astrophysical Journal, Vol. XLIX, No. 2, March 1919.

Theory of Imperfect Gratings, by C. M. Sparrow. (The paper attempts to present the whole subject from the stand point of the vector-method and contains in addition a theory of the defective Fabry and Parot interferometor with consequences relating to the method of coincidences as a means of measuring wave-lengths). *Studies based on the Colours and Magnitudes in Stellar Clusters Tenth Paper : A critical Magnitude in the sequence of stellor Luminositics* by Harlow Shapley. (In the case of cluster-type variables, the absolute photographic magnitude —0·2 defines a limit of stellar luminosity at which the liability to Cepheid variation suddenly stops. This limit is probably coincident with a turning point in the development of the internal structure of a star. The question whether analogous dynamical changes occur at the same epoch of luminosity is also discussed). *Aerial Photometry*, by M. Luckish. (Certain photometric measurements and allied observations relating to Landscapes, Clouds, Haze, Sky and Water are presented, introduced by a few paragraphs dealing with the general characteristics of natural lighting).

The American Mathematical Monthly, Vol. XXVI, No. 3, March 1919.

On the Envelope of the Wallace Lines of an inscribed Quadrangle, by D. F. Barrow. (If we take a quadrangle inscribed in a circle and omit each vertex in turn, we obtain four triangles, and the feet of the perpendiculars from any point on the circle upon the four Simpson lines with regard to these triangles lie on a line called the Wallace Line of the point with regard to the quadrangle. The envelope of the Wallace Lines is an involute of a four cusped hypocycloid whose cusps lie on a circle of radius three times that of the circumcircle. Other theorems are deduced and generalizations indicated),

Annals of Mathematics—Second Series, Vol. 20, No. 3, March 1919.

On Quaternions and their Generalization and the History of the Eight Square Theorem, by L. E. Dickson. (The object is to present the history of the generalizations to 4 and 8 squares of the formula, $(a^2+b^2)(\alpha^2+\beta^2)=r^2+s^2$, where $r=a\alpha-b\beta$, and $s=a\beta+b\alpha$ and an elementary exposition of Hurwitz's proof that such a formula holds only for 2, 4 or 8 squares. The writer shows that for these three cases, the formula admits of a simple interpretation concerning the norm of numbers which are ordinary complex numbers, quaternions or numbers of Cayley's algebra with 8 units. No knowledge of quaternions or the latter algebra is presupposed.) *Non-Symmetric Kernels of positive Type*, by Dr. Caroline E. Seely. (The paper considers properties of non-symmetric kernels, analogous to those of continuous symmetric functions of positive type, treated in Goursat, *Cours d'Analyse*, Vol. 3, p. 449). *Elementary Properties of the Stieltjes Integral*, by H. E. Bray. (Stieltjes Integral is defined and its existence proved. This is followed by other theorems leading up to a theorem on the change of order of integration of an iterated integral and another on integration by parts.) *A Kinematical Property of Ruled Surfaces*, by G. K. Whittemore. (The author proves that every ruled surface S (not developable) may be generated by a radius fixed in a sphere whose centre moves with unit velocity along the line of striction and which turns about the tangent to this curve from the binormal towards the principal normal with angular velocity equal to the reciprocal of the parameter of distribution, deduces two corollaries and the necessary and sufficient condition that the parameter of distribution may be equal to the radius of torsion of the line of striction at the corresponding points.) *Systems of Linear Inequalities* by Lloyd L. Dines. (The existence and character of the solution of a system of inequalities

$$\{\, a_{r1}\, x_1 + a_{r2}\, x_2 + \ldots\ldots + a_{rn}\, x_n > 0,\ r = 1, 2, 3, \ldots n \,\}$$

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PROGRESS REPORT.

The following gentlemen have been enrolled as *Life Members* of our Society :—

1. *Mr. Balak Ram*, M. A., I. C. S.—Sessions Judge, Larkhana ;
2. *Mr. C. Bhaskaraiya*, M. A.—Assist. Prof of Physics, Nizamia College, Hyderabad (Deccan).

2. The following gentlemen have been elected members of our Society :—

1. *Mr. T. N. Krishnaswami Iyer*, B. A.—Asst. Lecturer, C. D. College, Anantapur, (at concessional rate) ;
2. *Mr. N. Kuppuswami*, M. A.,—Lecturer, The Training College, Trivandrum ;
3. *Mr. H. Subrahmania Aiyar*, B. A —Tutor in Math., H. H. the Maharaja's College, Trivandrum ;
4. *Mr. M. V. Rama Krishnan*, B. A.—Mathematics Lecturer, C. M. S. College, 52 Sannadhi Street, Viraraghavapuram (Tinnevelly Bridge Station) ;
5. *Mr. David E. Reuben*, B A. (Cantab & Bombay), I. C. S., Cuttack ,
6. *Mr. G. A. Srinivasan*, M. A., L. T.—Asst. Prof. of Math., The Presidency College, Madras.

3. Goursat's *Mathematical Analysis*, Vol. II, Part. I., *Functions of a Complex Variable*—translated by Hedrick & Dunkel,—has been purchased for the Library.

Poona,
15th Oct. 1919.
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D. D. Kapadia,
Honorary Joint Secretary.

A Problem of Diophantine Approximation

By G. H. Hardy.

I have on several occasions found myself faced by the following problem, which appears to be one of considerable interest and difficulty. Suppose that a and θ are positive and $a>1$, and that (x) denotes the difference between x and the integer nearest to x. Then, *in what circumstances can it be true that*

$$(a^n\theta) \to 0, \tag{1}$$

when $n\to\infty$, *i.e., that*

$$a^n\theta = p_n + \varepsilon_n \tag{2}$$

where p_n *is an integer and* $\varepsilon_n\to 0$?

The general problem seems, as I said, to be one of great difficulty. There is however one case in which the answer is almost immediate, namely, that in which a is an *algebraic* number. We have in fact the following theorem:

Theorem A.—*Suppose that* a *is a real algebraic number greater than* 1, *the root of an irreducible equation*

$$k_0a^m + k_1\ a^{m-1} + \ldots\ldots + k_m = 0, \tag{3}$$

where $k_0, k_1, \ldots\ldots, k_m$ *are integers. Then, in order that numbers* θ *should exist which satisfy* (1), *it is necessary and sufficient that* $k_0=1$, *so that* a *is an algebraic integer, and that the moduli of all the roots of* (3), *other than* a *itself, should be less than* 1. *The numbers* θ *are then all rational in the corpus of* a; *and*

$$(a^n\theta) = O(b^n), \tag{4}$$

where b *is the numerically largest root of* (3), *other than* a *itself.*

We have

$$\frac{\theta}{1-ax} = \sum p_n\, x^n + \sum \varepsilon_n\, x^n = p(x) + \varepsilon(x),$$

say; and

$$\frac{\theta}{1-ax}(k_0 + k_1\, x + \ldots + k_m x^m) = \sum q_n x^n + \sum \zeta_n x^n, \tag{5}$$

where

$$q_n = k_0p_n + k_1p_{n-1} + \ldots\ldots + k_mp_{n-m},$$
$$\zeta_n = k_0\varepsilon_n + k_1\varepsilon_{n-1} + \ldots\ldots + k_m\varepsilon_{n-m},$$

p's and ε's with negative suffixes being regarded as equal to 0. The left hand side of (5) is a polynomial of degree $m-1$, so that

$$q_n+\zeta_n=0 \ (n>m).$$

As q_n is an integer, and $\zeta_n\to 0$, we must have

$$q_n=0,\ \zeta_n=0,$$

from a certain value of n onwards. Thus $p(x)$ and $\varepsilon(x)$ are rational functions, with the denominator

$$k_0+k_1x+\ldots\ldots+k_mx^m.$$

It is evident that $p(x)$ can be expressed in the form

$$p(x)=\mathrm{p}(x)+\frac{l_0+l_1x+\ldots\ldots+l_{m-1}x^{m-1}}{k_0+k_1x+\ldots\ldots+k_mx^m},$$

where $\mathrm{p}(x)$ is a polynomial and $l_0, l_1, \ldots\ldots$ are integers. From this it follows, in the first place, that $k_0=1$, so that a is an algebraic integer*,

Now, let us denote the roots of (3) by

$$a,\ a_1,\ a_2,\ldots\ldots,\ b_1,\ b_2,\ldots\ldots,$$

the a's having moduli not less than 1, and the b's moduli less than 1. Then

$$(6)\qquad p\ (x)=\mathrm{p}\ (x)+\frac{\mathrm{A}}{1-ax}+\sum\frac{\mathrm{A}_k}{1-a_kx}+\sum\frac{\mathrm{B}_k}{1-b_kx},$$

the A's and B's being rational in the corpus of a and none of them being zero. On the other hand $\varepsilon(x)$, since its coefficients tend to zero, can have no pole inside or on the circle $|x|=1$, and so

$$(7)\qquad \varepsilon(x)=\mathrm{e}(x)+\sum\frac{\mathrm{B}'_k}{1-b_kx},$$

where $\mathrm{e}(x)$ is a polynomial. From (6) and (7) we obtain

$$\frac{\theta}{1-ax}=\mathrm{p}(x)+\mathrm{e}(x)+\frac{\mathrm{A}}{1-ax}+\sum\frac{\mathrm{A}_k}{1-a_kx}+\sum\frac{\mathrm{B}_k+\mathrm{B}'_k}{1-b_kx},$$

so that

$$\mathrm{A}=\theta,\ \mathrm{A}_k=0,\ \mathrm{B}_k+\mathrm{B}'_k=0.$$

But we have already seen that no A_k can vanish. It follows that there can be no root of (3) of the type a_k.

* If

$$\frac{l_0+l_1x+\ldots\ldots+l_{m-1}x^{m-1}}{k_0+k_1x+\ldots\ldots+k_mx_m}$$

is expansible in a Taylor's series with integral coefficients, k_0 must be 1. For a proof of this well known proposition see, *e.g.*, P. Fatou, 'Series trigonometriques et series de Taylor', *Acta Mathematica*, vol. 30, 1906, pp. 335-400 (p. 369).

The conditions stated in Theorem A, with reference to the equation (3), are therefore necessary. Also $\Theta = A$ is rational in the corpus of a, and ε_n, which is, from a certain value of n onwards, equal to $\Sigma B'_k b_k^n$, is of the form $O(b^n)$, where b is the numerically greatest root of (3), other than a itself.

In order to complete the proof of the theorem, it is only necessary to show that, if (3) satisfies the conditions stated, Θ's exist which satisfy (1). And this is obvious; for

$$a^n + \Sigma b_k^n$$

is integral for all values of n so that

$$(a^n) = O(b^n)$$

Thus (1) is satisfied when $\Theta = 1$.

2. Theorem A is a special case of a more general theorem.

THEOREM B. *Suppose that* $a_j (j = 1, 2, \ldots\ldots, r)$ *is an algebraic number, greater than* 1 *in absolute value, and the root of an irreducible equation*

$$k_{0,j} a_j^{mj} + k_{1,j} a_j^{mj-1} + \ldots\ldots + k_{mj,j} = 0; \tag{8}$$

that $P_j(n)$ *is a polynomial with integral coefficients; and that*

$$\phi(n, \Theta) = P_1(n) a_1^n \Theta_1 + P_2(n) a_2^n \Theta_2 + \ldots\ldots + P_r(n) a_r^n \Theta_r.$$

Then, in order that it should be possible to find a system of numbers $\Theta_1, \Theta_2, \ldots\ldots, \Theta_r$, *all different from zero, for which*

$$(\phi(n, \Theta)) \to 0,$$

when $n \to \infty$, *it is necessary and sufficient that* $k_{0,j} = 1$ *for each value of* j, *so that* a_j *is an algebraic integer, and that, among the complete system of roots of all the equations* (8), *all save* $a_1, a_2, \ldots\ldots, a_r$ *themselves should be in absolute value less than* 1. *Each* Θ_j *is then rational in the corpus of the corresponding* a_j, *and*

$$(\phi(n, \Theta)) = O(n^c b^n),$$

where c *is a constant and* b *is the numerically greatest among the roots of the equations* (8), *other than* $a_1, a_2, \ldots\ldots, a_r$ *themselves.*

The proof of this theorem does not differ in principle from that of Theorem A, the role of the polynomial

$$k_0 + k_1 x + \ldots\ldots + k_m x^m$$

being now played by

$$K(x) = \prod_{j=1}^{r} (k_{0,j} + k_{1,j} x + \ldots\ldots + k_{mj,j} x^{mj}) \, S_{j+1}$$

where s_j is the degree of $P_j(n)$. The special theorem is, from our present point of view, more interesting than the general one, and I shall therefore confine myself to enunciating the latter.

3. With Theorem A should be associated another theorem which is merely a special case of a theorem already proved by Borel**.

THEOREM C. *If a and Θ are positive, and $a>1$, and*

$$(a^n\Theta)=O(b^n),$$

where $0<b<1$, then a is algebraic.

Continuing Theorems A and C we obtain an interesting criterion for the transcendentality of a, viz.

THEOREM D. *If $a>1$, $\Theta>0$, and*

$$(a^n\Theta)\to 0,$$

then a is algebraic or transcendental according as the equation

$$(a^n\Theta)=O(e^{-\delta n})$$

is or is not true for some positive value of δ.

4. It is interesting to consider in more detail the two simplest cases of Theorem A.

In the first place, suppose $m=1$, so that a is rational. If then (1) is satisfied, a must be an integer, and $a^n\Theta$ must be an integer for sufficiently large values of n. Thus the only solutions are given by

$$\Theta=p/a^m,$$

where p is an integer.†

The next case is that in which $m=2$. Then the equation satisfied by a is of the form

** E. Borel, 'Sur une application d'un theoreme de M. Hadamard', *Bulletin de la Societe Mathematique de France*, ser. 2, vol. 18, 1894, pp. 22–25. Borel's theorem is really (when stated in slightly different language) that which corresponds to B as C corresponds to A.

The criterion is a simple one: to find an application of it is quite another matter; and I know of no example of a transcendental a which satisfies (1) for any value of Θ.

† This trivial case of the theorem is of interest in connection with Weierstrass's function. [See G. H. Hardy, 'Weierstrass's non-differentiable function', *Transactions of the American Mathematical Society*, vol. 17, 1916, pp. 301-305.]

$$x^2-mx-n=0;$$

and it will easily be verified that, in order that this equation should have two real roots a and b, such that $a>1$, $-1<b<1$, it is necessary and sufficient that

$$m\geq 1, \quad 1-m<n<1+m.$$

The simplest case is that in which $m=1$, $n=1$, and

$$a=\tfrac{1}{2}(\sqrt{5}+1), \quad b=\tfrac{1}{2}(\sqrt{5}-1).$$

The determination of the corresponding values of Θ presents no difficulty. We have in fact

THEOREM E. *If a is a real quadratic surd greater than* 1, *and values of* Θ *exist which satisfy* (1), *then*

$$a^2-ma-n=0,$$

where $m\geq 1$, $1-m<n<1+m$. *The corresponding values of* Θ *are the numbers*

$$\left\{p+\frac{q}{\sqrt{(m^2+4n)}}\right\}a^{-r}$$

if m is odd, and the numbers

$$\left\{p+\frac{q}{\sqrt{(\frac{1}{4}m^2+n)}}\right\}a^{-r}$$

if m is even: here r is zero or a positive integer, and p and q halves of integers; and, when m is odd, p and q are either both integers or both halves of odd integers.

When $a=\frac{1}{2}(\sqrt{5}+1)$ the simplest values of Θ are

$$\Theta=1, \Theta=\sqrt{5}.$$

A general Theorem relating to the Cartesian Oval.

By A. C. L. Wilkinson.

1. The co-ordinates of any point on a Cartesian Oval are expressed in terms of a parameter or by the equations (Greenhill, Elliptic Functions, § 236).

$$x+iy=\mathrm{sn}^2(u+iv)=\mathrm{sn}^2 w,$$

$$x-iy=\mathrm{sn}^2(u-iv)=\mathrm{sn}^2 w'.$$

When v is real, the real points on the two ovals are got by giving u real values and real values $+iK'$.

Theorem I, *If four points on a Cartesian Oval are concyclic, then*

$$u_1+u_2+u_3+u_4\equiv 0,\ modd\ (2K, 2iK').$$

For, consider the expression,

$$\begin{vmatrix} \mathrm{sn}^2 w_1 sn^2 w_1', & \mathrm{sn}^2 w_1, & \mathrm{sn}^2 w_1', & 1 \\ \mathrm{sn}^2 w_2 sn' w_2', & \mathrm{sn}^2 w_2, & \mathrm{sn}^2 w_2', & 1 \\ \mathrm{sn}^2 w_3 sn^2 w_3', & \mathrm{sn}^2 w_3, & \mathrm{sn}^2 w_3', & 1 \\ \mathrm{sn}^2 w_4 sn^2 w', & \mathrm{sn}^2 w_4, & \mathrm{sn}^2 w_4', & 1 \end{vmatrix} = 0.$$

By use of the addition formulæ this becomes

$$\left| (s_1^2-s^2)^2,\ (s_1cd+sc_1d_1)^2,\ (s_1cd-sc_1d_1)^2,\ 1-k^2s_1^2s^2 \right| = 0$$

where only the first row of the determinant is written,

and $s_1=\mathrm{sn}u_1$, $s=\mathrm{sn}\ iv$ etc., and this reduces to

$$\left| s_1^4,\ s_1^2,\ s_1c_1d_1,\ 1 \right| = 0,$$

which write,

$$\left| 1, \frac{1}{s_1^2}, \frac{1}{s_1^4}, \frac{c_1d_1}{s_1^3} \right| = 0,$$

or,

$$\left| 1, P(u_1), {}^2P(u_1), P^1(u_1) \right| = 0,$$

This gives

$$\sigma(u_1-u_2)\sigma(u_1-u_3)\sigma(u_1-u_4)\sigma(u_2-u_3)\sigma(u_2-u_4)\sigma(u_3-u_4)$$

$$\times\sigma(u_1+u_2)\ \sigma(u_3+u_4)=0$$

• $\mathrm{P}(u)$ denotes Weierstrassian function.

(Tannery and Molk· *Fonctions Elliptiques* Vol. III, pp 88, 89 ; Whittaker and Watson, *Analysis* pp 451, Ex. 22).

Whence $u_1+u_2+u_3+u_4\equiv 0$, modd $(2K, 2iK^1)$.

For the generating circles which touch the Oval at twc points, we have $u+u'\equiv 0$ modd (K, iK^1),

Giving four sets of circles corresponding to

$$u+u'=O, K, iK', K+iK',$$

of which the first is the axial system.

Cor. 1. If a generating circle touch the oval at P,Q and any circle through P,Q cuts the oval again in P',Q' then P',Q' lie on another generating circle of the same system.

Cor. 2. If a circle cuts the oval in four points P,Q,R,S and P',Q',R',S' be the other points of contact of generating circles of the same system touching the oval at P,Q,R,S, then P',Q',R',S' are concyclic.

Cor. 3. Let the three non-axial generating circles which touch a Cartesian Oval at P touch again in P_1, P_2, P_3 ; then the circle through $P_1P_2P_3$ cuts the oval in the image in the axis of the point where the circle of curvature at P cuts the oval.

For, P_1, P_2, P_3 are $K-u$, $iK'-u$, $K+i\dot{K}'-u$, whence the other point of intersection is $3u$.

2. Theorem II: *If a circle passes through two fixed points on a Cartesian Oval, the envelope of the chord joining the other two points of intersection is an axial circle having double contact with the oval and which touches the chord joining the fixed points.*

A special case of this theorem is when the chord joining the fixed points passes through a focus, in which case the other chord also passes through that focus, supposed not the triple focus. Let the sum of the arguments of the two fixed points be $-2a$. Then $a+u$, $a-u$ can be taken for the arguments of the other points of intersection of any circle through the two fixed points. The chord joining those points is

$$\begin{vmatrix} X+iY, & X-iY, & 1 \\ \mathrm{sn}^2(a+u+iv), & \mathrm{sn}^2(a+u-iv), & 1 \\ \mathrm{sn}^2(a-u+iv), & \mathrm{sn}^2(a-u-iv) & 1 \end{vmatrix} = 0.$$

writing $\alpha = a + iv$, $\beta = a - iv$, this is

$$X\{\operatorname{sn}^2(u+\beta) - \operatorname{sn}^2(u-\beta) + \operatorname{sn}^2(u-\alpha) - \operatorname{sn}^2 u(u+\alpha)\}$$

$$+iY\{\operatorname{sn}^2(u+\beta) - \operatorname{sn}^2(u-\beta) - \operatorname{sn}^2(u-\alpha) + \operatorname{sn}^2(u+\alpha)\}$$

$$+\operatorname{sn}^2(u+\alpha)\operatorname{sn}^2(u-\beta) - \operatorname{sn}^2(u-\alpha)\operatorname{sn}^2(u+\beta) = 0. \quad \ldots \quad (1)$$

If this touches a circle centre (X,Y) and radius p, we have the left hand side squared

$$= 4p^2\{\operatorname{sn}^2(u+\beta) - \operatorname{sn}^2(u-\beta)\}\{\operatorname{sn}^2(u-\alpha) - \operatorname{sn}^2(u+\alpha)\}$$

whence the left hand side of (1) must be equal to

$$\frac{\pm 8psscd\sqrt{-s_1s_2c_1c_2d_1d_2}}{(1-k^2s^2s_1^2)(1-k^2s^2s_2^2)}$$

Where s, s_1, s_2 are written for $\operatorname{sn}u$, $\operatorname{sn}\alpha$, $\operatorname{sn}\beta$ *etc.*

This being an identity and also a doubly periodic function which is constant, it has no *infinities.* Hence considering $u = \beta + iK'$ $u = \alpha + iK'$, we have

$$-X - iY + \frac{1}{k}\operatorname{sn}^2(\alpha+\beta) = 0,$$

$$X - iY - \frac{1}{k^2}\operatorname{sn}(\alpha+\beta) = 0;$$

whence $Y = 0$, $X = \frac{1}{k^2}\operatorname{sn}^2(\alpha+\beta)$.

Considering further the expansion in the neighbourhood of $u = \beta + iK'$ writing $u = x + iK'$ and using

$$\operatorname{sn}^2(u-\beta) = \frac{1}{k^2\operatorname{sn}^2 x} = \frac{1}{k^2}\left\{\frac{1}{x^2} + \frac{1}{3}(1+k^2) + P(x^2)\right\}$$

$$\operatorname{sn}^2(u+\gamma) = \frac{1}{k^2\operatorname{sn}^2(\beta+\gamma+x)} = \frac{1}{k^2}\left\{\operatorname{ns}^2(\beta+\gamma) - \frac{2\operatorname{cn}(\beta+\gamma)\operatorname{dn}(\beta+\gamma)}{\operatorname{sn}^2(\beta+\gamma)} + \ldots\right\}.$$

We have the further consideration of the nature of the infinity in the neighbourhood of $u = \beta + iK'$, the expanion

$$\frac{X}{k^2}\left\{-\frac{1}{x^2} + \text{finite terms}\right\} + \frac{1}{k^4}\left\{\frac{1}{x^2} + \frac{1}{3}(1+k^2) + \ldots\right\}$$

$$\times\left\{ns^2(\alpha+\beta) - \frac{2xcn(\alpha+\beta)\,dn(\alpha+\beta)}{sn^2(\alpha+\beta)} + \ldots\right\}$$

$$-\frac{1}{k^4}\left\{ \mathrm{ns}^2(\beta-\alpha)\mathrm{ns}^2 2\beta+\ldots. \right\}$$

$$=2p\left\{ -\frac{1}{k^2x^2}-\frac{1}{3}\,\frac{1+k^2}{k^2}+\ldots+\frac{1}{k^2}\mathrm{ns}^2(\alpha+\beta)+\ldots\ldots \right\}^{\frac{1}{2}}$$

$$\mathrm{X}\left\{ \frac{1}{k^2}\mathrm{ns}^2(\beta-\alpha)-\frac{1}{k^2}\mathrm{ns}^2(\alpha+\beta)+\ldots\ldots \right\}^{\frac{1}{2}}$$

Whence comparing the terms in

$$\frac{1}{x}=\frac{2\mathrm{cn}(\alpha+\beta)\mathrm{dn}(\alpha+\beta)}{k^4\mathrm{sn}^5(\alpha+\beta)}=\frac{2ip}{k^2}\left\{ \mathrm{ns}^4(\beta-\alpha)-\mathrm{ns}^4(\beta+\alpha) \right\}$$

Which gives the value of p in order that the point $u=\beta+iK'$ may not be an infinity of the function in question.

The function we are considering remains unaltered when we write $-\beta$ for β and when we interchange α and β and has thus no infinities in the fundamental parallelogram of period.

Having obtained the values of X, p a, direct verification can be given : we are required to prove

$$[-\mathrm{sn}^2(u+\beta)-\mathrm{sn}^2(u-\beta)+\mathrm{sn}^2(u-\alpha)-\mathrm{sn}^2(u+\alpha)$$

$$+k^2\,\mathrm{sn}^2(\alpha+\beta)\ \{\mathrm{sn}^2(u+\alpha)\,\mathrm{sn}^2(u-\beta)-\mathrm{sn}^2(u-\alpha)\,\mathrm{sn}^2(u+\beta)\}\,]^2$$

$$=\frac{4\,\mathrm{cn}^2(\alpha+\beta)\,\mathrm{dn}^2(\alpha+\beta)\,\mathrm{sn}^2(\alpha-\beta)}{\mathrm{sn}^2(\alpha+\beta)-\mathrm{sn}^2(\alpha-\beta)}$$

$$\{\mathrm{sn}^2(\alpha+\beta)-\mathrm{sn}^2(u-\beta)\}\ \{\mathrm{sn}^2(u+\alpha)-\mathrm{sn}^2(u-\alpha)\}$$

$$=\frac{16\,\mathrm{cn}^2(\alpha+\beta)\,\mathrm{dn}^2(\alpha+\beta)\,\mathrm{sn}^2(\alpha-\beta)\,\mathrm{sn}^2u\,\mathrm{cn}^2u\,\mathrm{dn}^2u\,(1-k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta)^2}{(1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2\beta)^2\,(1-k^2\,\mathrm{sn}^2u\,\mathrm{sn}^2\alpha)^2}$$

This reduces to establishing the identity

$$\frac{c_2l_2d_2}{(1-k^2s^2s_2^2)^2}\cdot\frac{c_1l_1d_1}{(1-k^2s^2s_1^2)^2}$$

$$+\frac{k^2\mathrm{sn}^2(\alpha+\beta)\ \{s^2c_1d_1c_2d_2-s_1s_2c^2d^2\}\ \{s_1c_2d_2-s_2c_1d_1\}}{(1-k^2s^2s_1^2)^2(1-k^2s^2s_2^2)^2}$$

$$=\frac{\mathrm{cn}(\alpha+\beta)\mathrm{dn}(\alpha+\beta)\mathrm{sn}(\alpha-\beta)(1-k^2s_1^2s_2^2)}{(1-k^2s^2s_1^2)(1-k^2s^2s_2^2)}$$

where s_1, s_1 s_2 are written for sn u, snα, snβ etc.

Multiplying up by the denominator

$$(1-k^2s^2s_1^2)^2(1-k^2s^2s_2^2)^2$$

this reduces to three identities, namely the coeffs of s^3, s^4 and the constant terms. These are

$$\frac{s_1c_1d_1-s_2c_2d_1}{s_1c_2d_2-s_2c_1d_2}=k^2sn\alpha sn\beta sn^2(\alpha+\beta)-cn^2(\alpha+\beta)cn(\alpha+\beta)$$

$$sn^2(\alpha+\beta)\{c_1c_2d_2d_2+(1+k^2)s_1s_2\}=(s_1^2+s_2^2)cn(\alpha+\beta)dn(\alpha+\beta)+cs_1s_2.$$

$$\frac{s_1^2c_2d_2-s_2^2c_1d_1}{s_1c_2d_2-s_2c_1d_1}=sn^2(\alpha+\beta)-s_1s_2cn(\alpha+\beta)dn(\alpha+\beta)$$

The first is verified without much difficulty, or the first and third may be verified by solving for $sn^2(\alpha+\beta)$ and $cn(\alpha+\beta)dn(\alpha+\beta)$.

Substituting for α, β their values $a+iv$, $a-iv$, the circle we have obtained for the envelope has for its equation.

$$\left\{X-\frac{1}{k^2}\ ns^2 2a\right\}^2+Y^2=-\frac{cn^2 2a\,dn^2 2a\,sn^2 2iv}{k^4 sn^4 2a(cn^2 2a-sn^2 2iv)}.$$

Mr. V. Ramaswami Aiyar has given for the square of the radius of the axial circle centre G having double contact with the Cartesian Oval, the expression

$$\frac{GF_1.\ GF_2.\ GF_3}{GO}$$ (J. I. M. S. Vol. XI, p. 138.)

Taking G as $\frac{1}{k^2}ns^2 2a$, we have in case G lies between F_1F_2,

$$F_1G=\frac{1}{k^2}ns^2 2a,$$

$$GF_2=1-\frac{1}{k^2}\ ns^2 2a\ =-\frac{1}{k^2}ds^2 2a,$$

$$GF_3=\frac{1}{k^2}-\frac{1}{k^2}ns^2 2a=\frac{1}{k^2}\ cs^2 2a,$$

$$OG=OF_1+F_1G\ =\ \frac{1}{k^2}\ ns^2 2iv.$$

The last result being obtained as follows:—From Greenhill. *Elliptic Functions*, § 236, we have if r_1, r_2, r_3 are the distances of P from F_1, F_2, F_3.

$$r_2+r_1\,\mathrm{dn}\,2iv = \mathrm{cn}\,2\,iv$$

$$r_1+r_2\,\mathrm{cn}\,2iv = \frac{1}{k^2}\mathrm{dn}2iv.$$

$$r_3\,\mathrm{dn}2iv - r_2\mathrm{cn}2iv = \frac{k'^2}{k^2}.$$

Comparing these with Mr. Ramaswami Aiyar's equations (J. I. M. S. Vol. XI, p. 124 (2)), we find

$$\mathrm{cn}\,2iv = \sqrt{(c/a)}\,\mathrm{dn}2iv = \sqrt{(b/a)},\ b-a=1,\ c-a=k^{-2};$$

whence $\mathrm{sn}\,2iv = \frac{i}{kvu}$ and $a = OF_1 = -\frac{1}{k^2}\,\mathrm{ns}^2\,2iv.$

Our circle is thus identified with the circle centre

$$\left(\frac{1}{k^2}\,\mathrm{sn}^2\,2a,\ 0\right)$$

having double contact with the Cartesian Oval.

If P,Q, are the fixed points, P′,Q′, their images in the axis of the Oval, PQ P′Q′ are concyclic, hence the circle touches P′Q′ and thus also PQ.

As a particular case of this theorem we see that the 8 tangents to either of two confocal Cartesians which cut orthogonally at their 8 points of intersection touch a circle.

Multiplication of Infinite Integrals

By. K. B. Madhava, M.A.

1. Introductory.

Among the fundamental theorems relating to infinite series that are absolutely convergent is the theorem which states that the sum of such a series is not altered by any kind of derangement of its terms.* This theorem finds nseful application in the multiplication of two absolutely convergent series

$$U = u_1 + u_2 + \ldots + u_n + \ldots ad\ inf$$

$$V = v_1 + v_2 + v_3 + \ldots + \ldots ad\ inf$$

According to what is known as Cauchy's rule of writing the product, the typical term of the product series is

$$u_n v_1 + u_{n-1} v_2 + \ldots + u_2 v_{n-1} + u_1 v_n$$

and is written for shortness sake w_n. When the series $\sum_1^\infty w_n$ is convergent. say to W, the series Σw_n is said to be the product of the two series $\Sigma u_n, \Sigma v_n$ and $W = U.V$. Though according to Hardy† "the last word has not yet been said upon the subject" of the multiplication of infinite series and although "a number of interesting questions connected with it remain unanswered, a good many results of a far-reaching character have been obtained from it, the more recent of them being due to Hardy himself. In the present paper we propose to give an account of the analogous theory of the multiplication of infinite integrals according to what we may call "Cauchy's rule". This theory is naturally allied to that of infinite series, although the analogy between such integrals and absolutely convergent series is not quite complete, as there is no *order* in the values of a function. The problem of this paper is therefore similar though not simple; and the circumstance that no account of this, however brief is given in any book‡ that the writer has come across has been an additional inducement to write this paper which owes not a little to the memoir of Hardy just cited. In order to clearly see the parallelism between the integrals and the infinite series, and also to have an exact idea of the present state of our knowledge concerning the multiplication of such series, we have found it convenient to collect in the next paragraph the results that have been obtained in this connection.

* Bromwich: *Infinite Series*. P. 65.

† *Proc. Lond. Math. Soc.* Ser. 2. Vol. 6. p. 410.

‡ See, however, Bromwich. *loc. cit.* App. III.

	If it is known in advance that						And in addition that
	U		V		W		
1°	Convergent	...	Convergent	...	Convergent	...	
2°	Abs. Conv.	...	Abs. Conv,	...			
3°	Abs. Conv.	...	Convergent	...			
4°	Abs. Conv.	...	Oscillating between V_1 and V_2				$v_n \longrightarrow 0$ (an important necesssary condition).
5°	Convergent	...	Convergent	...			$w_n \longrightarrow 0$ (may also be replaced by other couditions).
6°	Convergent	...	Convergent	...			$\times$...
7°	Convergent	...	Convergent	...			
8°	Convergent	...	Convergent	...			$nu_n \to 0$; $nv_n \to 0$
9°	Convergent	...	Convergent	...	(K and L consts).		$\lvert nu_n \rvert \angle$ K ; $\lvert uv_n \rvert \angle$ L
10°	Convergent	...	Convergent	...			$n\psi(n)u_n \longrightarrow 0$ $nv_n/\psi(n) \longrightarrow 0$

cation of infinite series.

	Has proved one resnlt.	References.	Remarks.
Abel ...	W = UV ...	[Bromwich : p. 83.	[*Oeuvres* (1), p. 226]
Cauchy ...	W (is abs. conv.) = UV.	[Bromwich : p. 83.	[Bromwich : p. 95. *Jour. Fur. Math.* 79 (1875) p. 182.
Mertens ...	W (is conv.) = UV ...		This includes 2°.
Hardy—Mertens ...	W oscillates between UV_1 and UV_2 ...	[*Proc. Lond. Math Soc* (II) Vol. 6. p. 420.]	If $V_1 = V_2$ this reduces to 3.
Pringsheim...	$\Sigma(-)^n w_n$ is conv. (U and V are also alternate series).	[Bromwich : p. 86.	All these are such that one at least of U and V be-becomes abs. conv.
Voss ...		[*Math Ann.* 24 (1884) p. 42.	
Cajori ...		*Am. Jour of Math.* 15 (1893), p. 339. Also *N. Y. Bulletin* 1895 (pp. 180)].	
Hardy (I. Th.) ...	W (is conv.) = UV ...	[*Proc. Lond Math Soc.* (II) Vol. 6 p. 410 to 428.]	Hardy suggests (as one of the unsolved problems) the construction of an example where $u_n = \pm n^{-s}$ $v_n = \pm n^{-t}$ $s+t>1$, but one at least $\angle 1$, so that U and V are convergent but W is oscillating, for a proper choice of signs.
Hardy (III. Th.) ...	Do.		
Hardy (II. Th.) ...	$\psi(n) = (\log n)^{\alpha}$ (log log n)$^{\beta}$ any function of the class of which log n is typical.		

The proofs of the foregoing theorems all hinge upon the fact that Σw_n cannot diverge (although it may oscillate) when Σu_n and Σv_n are convergent, while the machinery for the proofs, stripped of the formal algebraic reductions, reduces to the application of two important lemma : viz, *Abel's partial summation lemma*, and *Cesaro's mean value theorem* in one or other of its forms.

Of these theorems, 4°, 5°, 6° and 7° reduce themselves to conditions for the convergence of integrals which involve a periodic factor (or a monotonic non-increasing factor) of which a variety of forms are well known.*

The theorems 1°, 2°, 3°, 8°, 9°, 10° on the other hand, have proper analogues in the theory of the multiplication of infinite *integrals* and these will be set out in the subsequent paragraphs.

III. Notation and Preliminary theorems.

We shall denote by $u\,(x)$ and $\mathrm{v}\,(x)$ two continuous functions of x, by $u_1\,(x)$ and $v_1\,(x)$ their absolute values. Let us suppose that the functions are such that $\int_0^\infty u\,(x)\,dx$ converges to U, (3·1)

and $\int_0^\infty \mathrm{v}\,(x)\,dx$ converges to V. (3·2)

If $u\,(x)$ and $\mathrm{v}\,(x)$ in the integrands can be replaced by $u_1\,(x)$ and $v_1\,(x)$, the integrals are absolutely convergent.

Let us form now by combining u, and v a typical function $w\,(x)$ defined by

$$w(x) = \int_0^x u(\mathrm{y})\ \mathrm{v}(x-y)\,dy, \tag{3·3}$$

which is also seen after the substitution $x-y$, for y, to be equal to

$$\int_0^x u(x-y)\mathrm{v}(y)dy. \tag{3.31}$$

* See, for instance Bromwich : § 169, and various papers by Young, Hobson etc among English writers.

This method of defining $w(x)$ coresponds to taking—$w_n = u_1 v_n + u_2 v_{n-1} + \ldots + u_n v_1$, (in which u and v are easily seen to be reversible) in Cauchy's method of multiplying two series (or adding by diagonals a double series).

Let us also employ the notation

$$\left.\begin{aligned} \mathrm{U}(x) &= \int_0^x u(y)dy, \\ \mathrm{V}(x) &= \int_0^x v(y)dy, \\ \text{and } \mathrm{W}(x) &= \int_0^x w(y)dy\ ; \end{aligned}\right\} \ldots \qquad 3.4)$$

analogous to $\mathrm{U}_n = \Sigma u_n$, etc ; in the other theory.

Then we have the result that

$$\mathrm{W}(x) = \int_0^x u(y)\mathrm{V}(x-y)dy, \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}5)$$

or the other equivalent forms

$$\int_0^x u(x-y)\mathrm{V}(y)dy,$$

$$\int_0^x \mathrm{U}(x-y)v(y)dy,$$

$$\int_0^x \mathrm{U}(y)v(x-y)dy,$$

[This coresponds to putting.

$$\mathrm{W}_n = u_1\mathrm{V}_n + u_2\mathrm{V}_{n-1} + \ldots + u_n\mathrm{V}';$$
$$= \mathrm{U}_1 v_n + \mathrm{U}_2 v_{n-1} + \ldots + \mathrm{U}_n v_1.]$$

For, $\dfrac{d\mathrm{W}}{dx} = w\ (x) = \int_0^x u(y)v\ (x-y)\ dy,$

$$= U(y)v(x-y)\Big]_0^x + \int_0^x U(y)v'(x-y)dy,$$

(on integrating by parts)

$$= U(x)v(0) + \int_0^x U(y)v'(x-y)dy$$

Hence $W(x) = \int_0^x U(y)\, v(x-y)\ dy.$

This we may call "Abel's transformation." Of course we could also have established this directly by transforming the double integrals.

The problem now is to consider the integral

$$\int_0^\infty w(x)dx, \qquad (3\cdot61)$$

or (written in its equivalent form)

$$\int_0^\infty dx \int_0^x u(y)v(x-y)dy \qquad (3\cdot62)$$

or δ the limit (if it exists) of,

$$\lim_{x \longrightarrow \infty} \frac{1}{x} \int_0^x dy \int_0^y dz \int_0^z u(t)v(z-t)dt$$

in its relation to the product of (3·1) and (3·2) (3·63)

viz. $\int_0^\infty u(x)dx$ and $\int_0^\infty v(x)dx.$

In other words, the problem is: under what circumstances (in addition, of course to U and V being convergent), is the integral (3·6) convergent?, and when convergent what is its value in terms of U and V?

The problem being thus put, we proceed to obtain some preliminary lemmas.

First, we have

$$\lim_{x \to \infty} \frac{1}{x} \int_0^\infty U(y).\ V(x-y)dy = U.\ V \ldots\ldots \quad (3.7)$$

Since the integral giving U is convergent, we can choose a number such that for all values of greater than the number $\cdot U-U(x)$ shall be less than ε_1 where ε_1 is an arbitrarily small preassigned positive number. Likewise let x_2 and ε_2 refer to V(x). Then choosing x greater than either of these values (say $x>x_3$) we shall have

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x U(y)V(x-y)dy$$

$$=\lim_{x\to\infty}\frac{1}{x}\int_0^x (U+\varepsilon_1)(V+\varepsilon_2)\,dy \quad \text{at least when } x>x_3$$

$\to$ U. V. since $\varepsilon\to 0$ with $\frac{1}{x}$.

It is now easy to obtain a theorem corresponding to Cesaro's theorem $\frac{1}{n}(W_1+W_2+\ldots+W_n)=\frac{1}{n}(U_1V_n+U_2V_{n-1}+\ldots+U_nV_1)\to$ U V. This theorem connects (3·6) with (3·5) and (3·7) and is this, viz :

$$\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x dy\int_0^y dz\int_0^z dt\; u(t)v(z-t) \qquad \ldots \qquad \ldots \qquad (3\cdot63)$$

$$=\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x dy\int_0^y w(z)dz \qquad \ldots \text{ from } (3\cdot31)$$

$$=\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x W(y)\,dy, \qquad \ldots \text{ from } (3\cdot4)$$

$$=\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x dy\int_0^y u(z)V(y-z)dz, \text{ from } (3\,5)$$

$$=\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x U(y)V(x-y)\,dy$$ by repeating the method by which (3 5) was obtained.

ie to say $\longrightarrow$UV

Conseqnently,

$$\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x dy\int_0^y dz\int_0^z dt\,u(t)v(z-t)\longrightarrow UV \qquad \ldots (3.\ 8)$$

which is "the analogue of Cesaro's theorem."

The preliminaries being put thus, we shall obtain in the next paragraph the analogue of Abel's theorem : 1° of § II.

IV : The analogue of Abel's theorem.

The theorem is, that when

$$\int_0^\infty u(x)dx,\ \int_0^\infty u(x)dx \text{ and } \int_0^\infty w(x)dx \left[(3\cdot1),\ (3\cdot2) \text{ and } (3\cdot6)\right]$$

are convergent, the last mentioned value is equivalent to the product of the first two; that is to say $W = UV$.

This follows immediately from § III, for,

Since $\int_0^\infty w(x)dz$ is convergent and equal to W.

$$\lim_{x\to\infty} W(x) \text{ exists and equals W;}$$

or $$\lim_{x\to\infty} \frac{1}{x}\int_0^x W(y)dy \text{ also exists and is equal to W.}$$

$$\text{Hence } W = \lim_{x\to\infty} \frac{1}{x}\int_0^x W(y)dy$$

$$= \lim_{x\to\infty} \frac{1}{x}\int_0^x dy \int_0^y dz \int_0^x u(t)v(z-t)dt \text{ as in } (3\cdot8)$$

$$= UV \text{ from } (3\cdot8).$$

Thus we have,

$$\int_0^\infty u(x)dx \times \int_0^\infty v(x)dx = \int_0^\infty dx \int_0^x u(y)v(x-y)dy \qquad (4)$$

when all the three integrals are convergent.

As illustrations, we have

1 $$U = \int e^{-2x} x^{2n-1} dx = \frac{\Gamma(n)}{2^n}$$

$$\text{and } V = \int_0^\infty e^{-x} dx = 1$$

$$\text{Hence, } W = \int_0^\infty dx \int_0^x x^{n-1} e^{-x-y} dy = \frac{\Gamma(n)}{2^n} \quad (4\cdot1)$$

2. $$\int_0^\infty dx \int_0^x \frac{[\sin by + \sin b(2x-y)]}{x(x-y)} \exp(-a^2x^2)dy$$

$$= \frac{\pi^{2/3}}{2a} \exp\left[-\frac{b^2}{a^2}\right]$$

This theorem enables us to evaluate some double integrals which cannot be integrated (otherwise) by direct methods.

*Bromwich p. 83 and p, 385,

A Proof of Bertrand's Postulate

By S. Ramanujam.

1. Landau in his *Handbuch* pp. 89-92 gives a proof of a theorem the truth of which was conjectured by Bertrand : namely that there is at least one prime p such that $x<p\leq 2x$, if $x\geq 1$. Landau's proof is substantially the same as that given by Tchebicheff. The following is a much simpler one :

Let $\nu(x)$ denote the sum of the logarithms of all the primes not exceeding x and let

(1) $\psi(x)=\nu(x)+\nu(\sqrt{x})+\nu(x^{\frac{1}{3}})+\ldots\ldots\ldots,$

(2) $\log[x]!=\psi(x)+\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)+\ldots\ldots$, where $[x]$ denotes as usual the greatest integer in x.

From (1) we have

(3) $\psi(x)-2\psi(\sqrt{x})=\nu(x)-\nu(\sqrt{x})+\nu(x^{\frac{1}{3}})-\ldots\ldots\ldots$

and from (2).

(4) $\log[x]!-2\log[\frac{1}{2}x]!=\psi(x)-\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)-\ldots\ldots.$

Now remembering that $\nu(x)$ and $\psi(x)$ are steadily increasing functions, we find from (3) and (4) that

(5) $\psi(x)-2\psi(\sqrt{x})\leq\nu(x)\leq\psi(x)$;

and (6) $\psi(x)-\psi\left(\frac{x}{2}\right)\leq\log[x]!-2\log\left[\frac{x}{2}\right]!\leq\psi(x)-\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)$

But it is easy to see that

(7) $\log\Gamma(x)-2\log\Gamma\left(\frac{x+1}{2}\right)\leq\log[x]!-2\log\left[\frac{x}{2}\right]!$

$$\leq\log\Gamma(x+1)-2\log\Gamma\left(\frac{x+1}{2}\right).$$

Now using Stirling's approximation we deduce from (7) that

(8) $\log[x]!-2\log[\frac{1}{2}x]!<\frac{3}{4}x$, if $x>0$; and.

(9) $\log[x]!-2\log[\frac{1}{3}x]!>\frac{2}{3}x$, if $x>300$ at any rate.

It follows from (6) (8) and (9) that

(10) $\psi(x)-\psi(\frac{1}{2}x)<\frac{3}{4}x$, if $x>0$;

and (11) $\psi(x)-\psi\left(\frac{x}{2}\right)+\psi\left(\frac{x}{3}\right)>\frac{2}{3}x$, if $x>300$.

Now changing x to $\frac{1}{2}x$, $\frac{1}{4}x$, $\frac{1}{8}x$,.....in (10) and adding up all the results, we obtain

(12) $\psi(x) < \frac{3}{2}x$, if $x > 0$.

Again we have

$$(13)\quad \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \leq \nu(x) + 2\psi(\sqrt{x}) - \nu\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right)$$

$$< \nu(x) - \nu\left(\frac{x}{6}\right) + \tfrac{1}{2}x + 3\sqrt{x},$$

in virtue of (5) and (12).

It follows from (11) and (13) that

(14) $\nu(x) - \nu\left(\frac{x}{2}\right) > \frac{x}{6} - 3\sqrt{x}$, if $x > 300$. But it is obvious that

$$\frac{x}{6} - 3\sqrt{x} > 0 \text{ if } x \geq 324.$$

Hence

(15) $\nu(2x) - \nu(x) > 0$ if $x \geq 162$.

In other words there is at least one prime between x and $2x$ if $x \geq 162$. Thus Bertrand's Postulate is proved for all values of x not less than 162; and, by actual verification we find that it is true for smaller values.

2. Let $\pi(x)$ denote the number of primes not exceeding x. Then since $\pi(x) - \pi(\frac{1}{2}x)$ is the number of primes between x and $\frac{1}{2}x$, and $\nu(x) - \nu(\frac{1}{2}x)$ is the sum of logarithms of primes between x and $\frac{1}{2}x$ it is obvious that

(16) $\nu(x) - \nu(\frac{1}{2}x) \geq \{\pi(x) - \pi(\frac{1}{2}x)\} \log x$, for all values of x. It follows from (14) and (16) that

(17) $\pi(x) - \pi(\frac{1}{2}x) > \frac{1}{\log x}\left(\frac{x}{6} - 3\sqrt{x}\right)$ if $x > 300$. From this we easily deduce that

(18) $\pi(x) - \pi(\frac{1}{2}x) \geq 1, 2, 3, 4, 5, \ldots\ldots$ if $x \geq 2, 11, 17, 29, 41, \ldots\ldots$ respectively.

Astronomical Notes.

Comets.

1. Kopff's periodic comet (1906 IV) which returned to perihelion on 1919 June 23, was discovered on July 30 by Prof. Max. Wolf at Heidelberg. Its stellar magnitude was then about 10. 5. The comet was not visible during its last return 1912 Nov. 30 as its position, relative to the Sun was then not so favourable.

2. The second Comet for the year (1919 b) was discovered by Metcalf in America on August 20. At the time, when the discovery was announced, the comet was supposed to be new and the following elements for a parabolic orbit were computed from the available observations :—

$$T = 1919 \text{ Oct. } 16.\ 1984 \text{ Greenwich Mean Time.}$$

$$\left.\begin{aligned} \omega &= 128^\circ\ \ 33'.\ 32 \\ \Omega &= 311\ \ 22.\ 93 \\ i &= 19\ \ 58.\ 03 \end{aligned}\right\} \quad 1919.\ 0$$

$$\log q = 9.\ 6828$$

Prof. Leuschner has subsequently identified it with Comet 1847 V (Brorsen), a member of the Neptune group, the actual period being 72 years 37 days. The magnitude during these months has been ranging between 6 and 7, so that the Comet was just visible to the naked eye and could easily be seen through a small telescope. Its position on Oct. 10 was R. A. $11^h\ 46^m$, Dec. 16°. in the eastern sky early morning.

3. News has also been received of the discovery of another Comet (1919 c) by Metcalf on August 22. It was also discovered independently by Borelly, a day later in Europe. The Comet was faint at the time of discovery being about the tenth magnitude. Some subsequent observations have also been obtained, but sufficient details are not forthcoming to show that the Comet is likely to prove interesting.

Planets.

4. Most of the planets are morning stars. Their positions on Nov. 1 are as follows :—

	R. A.	Decl.	Constellation
Mercury	15^h 42^m	22° .0 S.	Scorpio
Venus	11 35	2 .0 N.	Virgo
Mars	11 1	7 .9 N.	Leo
Jupiter	9 16	16 .5 N.	Cancer
Saturn	10 46	9 .5 N.	Leo
Uranus	22 1	13 .0 S.	Pisces
Neptune	8 56	17 .3 N.	Cancer.

The position of the Sun on the same day will be R. A. 14^h 23^m, Decl. 4° 11′ S. Venus attains its greatest brilliancy on Oct. 20 and will be at greatest elongation on Nov. 23.

Eclipses.

5. A partial Eclipse of the Moon will occur on Nov. 8, 1919. The beginning of the Eclipse will be visible throughout India but the end will not be visible, as the Moon sets before leaving the penumbra. The circumstances of the Eclipse are given below :

Moon enters Penumbra	Nov. 8	3^h 4^m	A. M.	}	Indian Standard Time.
Moon enters Umbra	„ „	4 28	„		
Middle of the Eclipse	„ „	5 14	„		
Moon leaves Umbra	„ „	6 0	„		
Moon leaves Penumbra	„ „	7 25	„		

Also an annular eclipse of the Sun (invisible in India) will occur on Nov. 22. The central line begins in Texas U. S. A., passes through Cuba, and after crossing the Atlantic ends in the North West of Africa.

6. Some photographs have been secured during the recent total Solar Eclipse (May 28) by Prof. Eddington, with a view to test Einstein's Theory of Relativity. The results have not yet been fully discussed, but there seems to be some evidence of deflection in the positions of stars due to the Einstein effect.

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Hyderabad,

T. P. Bhaskara Sastri,

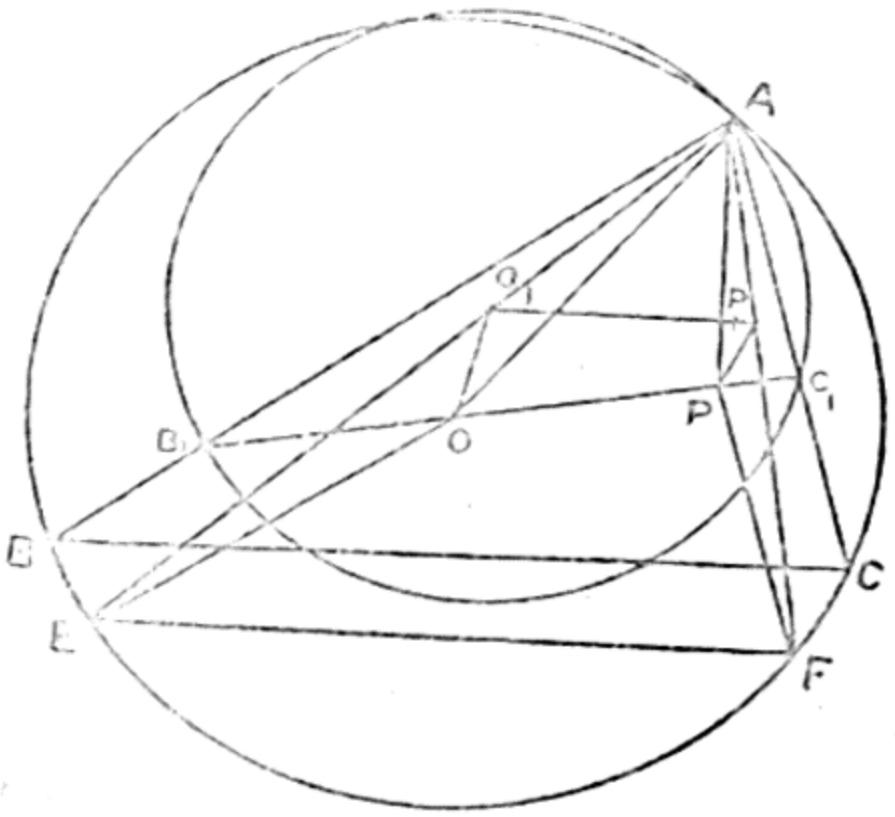
SOLUTIONS

Question 1009.

(Communicated by Mr. HEMRAJ) :—ABC is a triangle. O,P are its circumcentre and orthocentre. If OP meets AB, AC, in B_1C_1, prove that the join of the circumcentre and orthocentre of AB_1C_1 is parallel to BC.

Solution (1) *by T. Rajarama Roa and* (2) *by Lakshmishanker N. Bhatt.*

(1) Let P_1, O_1 be the orthocentre and the circumcentre of Δ AB_1C_1; Let AO_1; AP_1 produced meet the circle ABC in E,F. Draw OM; O_1M_1 perpendiculars on BC; B_1C_1. Join BO, B_1O_1, O_1O, OE; PP_1; PF, OA.



Then the triangles OBM; $O_1B_1M_1$ are similar.

$$\therefore \quad \frac{OB}{O_1B_1} = \frac{OM}{O_1M_1} = \frac{AP}{AP_1}.$$

But AO, AP and AO_1, AP_1 are pairs of isogonal conjugates, and $\angle OAO_1 = \angle PAP_1$.

It follows that Δ's; O_1AO and P_1AP are similar.

$$\therefore \quad \frac{AO_1}{AP_1} = \frac{AO}{AP}.$$

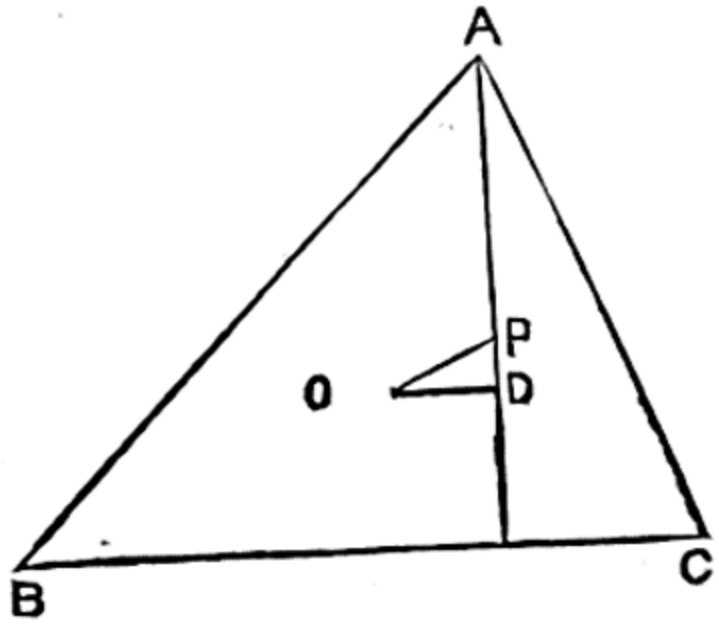
But PA = PF since OP is perpendicular to AF, a chord,

$\therefore$ the isosceles triangles AOE, APF are similar.

that is
$$\frac{AO_1}{AP} = \frac{AE}{AF} = \frac{AO_1}{AP_1};$$

$$\therefore \quad O_1P_1 \parallel EF \parallel BC.$$

(2) Join AP and draw OD $\perp$ to A.P. Then if OP makes an angle θ with BC or OD,



$$\tan\theta = \frac{PD}{OD} = \frac{R\cos(B-C)-2R\cos A}{c\cos B - \frac{1}{2}a}$$

$$= \frac{R\cos(B-C)-2R\cos A}{2R\cos B.\sin C - R\sin A} = \frac{\cos(B-C)-2\cos A}{\sin(C-B)} \quad \ldots \text{ (A).}$$

Now if θ_1 is the angle between OP or B_1C_1 and BC, A, $C+\theta$, $B-\theta$ are the angles of $\triangle AB_1C_1$;

and if the join of the orthocentre and circumcentre of this new triangle AB_1C_1, say O_1P_1, makes an angle θ with B_1C_1, we have similarly

$$\tan\theta_1 = \frac{(\cos B\text{-}\theta\text{-}\theta\text{-}C)-2\cos A}{\sin(\theta+C-B+\theta)} = \frac{\cos(2\theta-B+C)-2\cos A}{\sin(2\theta-B+C)}.$$

Hence if O_1P_1 is parallel to BC, $\theta+\theta_1$ must be equal to zero.

We therefore require

$$\frac{\cos(B-C)-2\cos A}{\sin(C-B)} + \frac{\cos(2\theta-B+C)-2\cos A}{\sin(2\theta-B+C)} = 0,$$

i.e. $\sin(2\theta-2B+2C) = 4\cos A\cos\theta\sin(\theta-B+C,)$

i.e. $2\sin(\theta-B+C)\cos(\theta-B+C) = 4\cos A\cos\theta\sin(\theta-B+C),$

i.e. $\cos(\theta-B+C) = 2\cos A\cos\theta,$

i.e. $\cos\theta\cos(B-C)+\sin\theta\sin(B-C) = 2\cos A\cos\theta,$

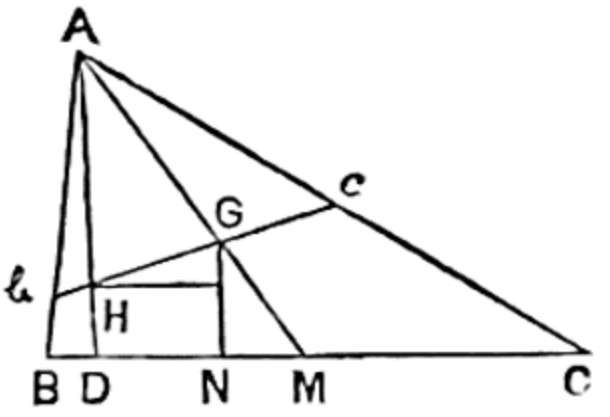
i.e. $\cos(B-C)-2\cos A = \tan\theta\sin(C-B),$.

But this is the case, from the value of tan θ found above in (A); hence the result is established.

Question 1037.

(A. C. L. Wilkinson):—If one angle of at riangle is 60°, prove that the Enler line of the triangle is equally inclined to the two sides containing the angle of 60°.

Solution by K. J. Sanjana.



Let $\angle A = 60°$; AD perpendicular to BC, H the orthocentre; AM the median to BC, G the centroid.

Let HG meet AB, AC in b, c respectively, and make an angle θ with BC.

Now $$\tan\theta = (GN - HD)/DN = (\tfrac{1}{3}AD - HD)/\tfrac{2}{3}DM$$
$$= (\tfrac{1}{3}AH - \tfrac{1}{3}HD)/\tfrac{2}{3}(BM - BD)$$
$$= \frac{2R\cos A - 4R\cos B\cos C}{R\sin A - 2R\sin C\cos B}$$
$$= \frac{2\cos A - \cos(B-C)}{\sin(B-C)} \quad \ldots\ldots(1)$$

If $A = 60°$, $2\cos A = 1$ and $\tan\theta = \tan\frac{1}{2}(B-C)$;

thus $\angle b = B - \theta = \frac{1}{2}(B+C) = 60°$, $\angle c = C + \theta = \frac{1}{2}(B+C) = 60°$.

This method also proves the converse; for, if $\angle b = \angle c$, $B - \theta = C + \theta$, and $\theta = \frac{1}{2}(B-C)$, so that $\tan\theta = \{1 - \cos(B-C)\}/\sin(B-C)$, Further, equation (1) gives a proof of Q. 1009, communicated by Prof. Hemraj.

Question 1038.

(A.C.L.Wilkinson).—The equation of a conic referred to tangential areal coordinates is

$$al^2 + bm^2 + cn^2 + 2hlm + 2fmn + 2gnl = 0;$$

show that the magnitudes of its semiaxes are given by the equation

$$I^2x^4 + 4IJR^2x^2 + 16R^4\Delta\sin^2A\sin^2B\sin^2C = o,$$

where R is the radius of the circumcircle of ABC, the triangle of reference, and

$$I = a + b + c + 2f + 2g + 2h,$$
$$J = A\sin^2A + B\sin^2B + C\sin^2C - 2F\cos A\sin B\sin C$$
$$- 2G\cos B\sin C\sin A - 2H\cos C\sin A\sin B,$$

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}, \qquad A = \frac{\partial\Delta}{\partial a},\ F = \tfrac{1}{2}\frac{\partial\Delta}{\partial f} \text{ etc.}$$

Hence, or otherwise, prove that if a conic is inscribed in a triangle the product of the squares of its semiaxes is $4\,R\,\alpha\beta\gamma$, where R is the radius of the circumcircle of the triangle and α, β, γ are the perpendiculars from the centre of the conic on the lines joining the middle points of the sides of the triangle.

Solution by K. J. Sanjana, M. A.

The notation has, I am afraid, got somewhat mixed: S and Δ are used indifferently for the same determinant, and a, b, c are liable to be confounded with the sides of the triangle ABC. I shall use the tangential areal form $ul^2+vm^2+wn^2+2u'mn+2v'nl+2w'lm=0$, which is more usual. The problem is solved in C. Smith's *Conic Sections* (New Ed. § 277), where however trilinear coordinates are used. The method may of course be extended to tangential areals; or we may obtain the semiaxes as the radii of circles whose centres are at the centre of the conic and which have double contact with it, the poles of the chords of contact being at infinity. The result may thus be obtained directly in tangential areal coordinates.

The formula arrived at by C. Smith is

$$\begin{vmatrix} \lambda U+x^2, & \lambda W'-x^2\cos C, & \lambda V'-x^2\cos B \\ \lambda W'-x^2\cos C, & \lambda V+x^2, & \lambda U'-x^2\cos A \\ \lambda V'-x^2\cos B, & \lambda U'-x^2\cos A, & \lambda W+x^2 \end{vmatrix}=0,$$

the *trilinear* equation of the conic being $\Sigma u\alpha^2+2\,\Sigma u'\beta\gamma=0$.

The coefficient of x^6 is zero, being $1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C$.

The coefficient of λx^4 is $U+V+W+2U'\cos A+2V'\cos B+2W'\cos C-U\cos^2 A-V\cos^2 B-W\cos^2 C+2U'\cos B\cos C+2V'\cos C\cos A+2W'\cos A\cos B=U\sin^2 A+V\sin^2 B+W\sin^2 C+2U'\sin B\sin C+2V'\sin C\sin A+2W'\sin A\sin B$.

If now u, v w w' v' w' refer to the *tangential* areal equation, we have to replace U by b^2c^2u,.....and U' by a^2bcu',.....

Thus for the coefficient of λx^4 we get

$b^2c^2u\sin^2 A+c^2a^2v\sin^2 B+a^2b^2w\sin^2 C+2a^2bcu'\sin B\sin C+2b^2cav'\sin C\sin A+2c^2abw'\sin A\sin B$

$$=4\Delta^2(u+v+w+2u'+2v'+2w')=4\Delta^2 I,$$

where Δ denotes as ordinarily the area of the triangle ABC.

The coefficient of $\lambda^3 x^3$ in the expanded determinant is

$$VW+WU+UV-U'^2+2\ UU'\cos A-V'^2+2\ VV'\cos B-W'^2+2\ WW'\cos C-2\ V'W'\cos A-2\ W'U'\cos B-2\ U'V'\cos C$$
$$=(VW-U'^2)+(WU-V'^2)+(UV-W'^2)$$
$$+2\cos A\,(UU'-V'W')+2\cos B\,(VV'-W'U')+2\cos C\,(WW'-U'V').$$

Changed to areal tangential notation, this is

$$\Sigma a^4b^2c^2\,(vw-u'^2)+2\,\Sigma a^2b^2c^2\cos A\,(uu'-v'w')$$
$$=a^2b^2c^2\{a^2U+b^2V+c^2W+2bc\cos A.U'+2ca\cos B.V'+2ab\cos C.W'\}$$
$$=4R^2a^2b^2c^2\{U\sin^2A+V\sin^2B+W\sin^2C+2U'\sin B\sin C\cos A+\ldots+\ldots\}$$
$$=4R^2a^2b^2c^2J.$$

[There is evidently a mistake of sign in the last three terms of the given value of J.]

Lastly, the coeft. of $\lambda^3 = UVW+2U'V'W'-UU'^2-VV'^2-WW'^2$;

$$=a^4b^4c^4\,(uvw+2u'v'w'-uu'^2-vv'^2-ww'^2)$$
$$=a^4b^4c^4\,D,\text{ where }D=\begin{vmatrix}u & w' & v'\\ w' & v & u'\\ v' & u' & w\end{vmatrix}.$$

Thus the equation to determine the lengths of the axes is

$$4\Delta^2 I x^4+4a^2b^2c^2R^2J\lambda x^2+a^4b^4c^4D\lambda^2=0.$$

Now in trilinear point coordinates $\lambda=4\Delta^2/\{\Sigma a^2U+2\Sigma bcU'\}$; hence in areal tangentials,

$$\lambda=4\Delta^2/\{\Sigma a^2b^2c^2u+2\Sigma a^2b^2c^2u'\}$$
$$=4\Delta^2/a^2b^2c^2\,I.$$

Therefore the required equation reduces to

$$4\Delta^2 I x^4+16\Delta^2R^2Jx^2/I+16\Delta^4D/I^2=0,$$

or

$$I^3x^4+4R^2IJx^2+4\Delta^2D=0,$$

which agrees with the result given.

When the conic is inscribed in the triangle of reference, the areal *tangential* equation is $u'mn+v'nl+w'lm=0$; also $2(u'+v'+w')\ 0=$ and $D=2u'v'w'$. The *equation* of the centre is $(v'+w')\ l=0$, so that its areal coordinates are as $v'+w' : w'+u' : u'+v'$, and their absolute values are $(v'+w')/I$, etc. The areal equation of the join of the mid-points of AB and AC is $y+z-x=0$; hence the distance of the centre from this line is given by

$$\alpha=2\Delta\cdot 2u'.I^{-1}\{a^2+b^2+c^2-2bc\cos A+2ca\cos B+2ab\cos C\}^{-\frac{1}{2}}$$
$$=2\Delta u'/aI.$$

So aslo β, γ may be found.

$$\therefore\ 4R\,\alpha\beta\gamma=4R.\frac{8\Delta^3u'v'w'}{abcI^3}=\frac{8\Delta^3u'v'w'}{I^3}=\frac{4\Delta^2D}{I^3};$$

and this is the product $x_1^2x_2^2$ from the equation obtained above.

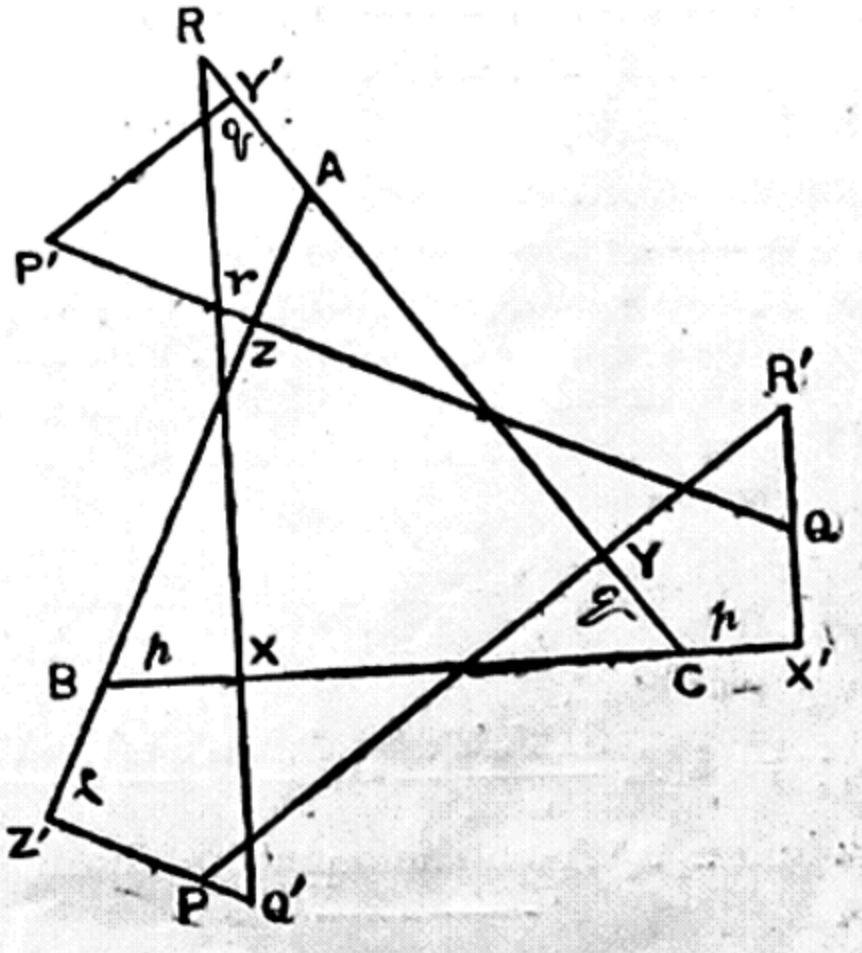
Questions 1043 & 1044.

1043 (V. Ramaswami Aiyar, M.A.) :—On the sides BC, CA, AB o a triangle ABC pairs of points (X, X′) Y, Y′), (Z,|Z′) are taken such that $XX' = BC$, $YY' = CA$ and $ZZ' = AB$; and perpendiculars are erected at these points to the sides. If the perpendiculars at (Y, Z′) (Z, X′), (X, Y′) interseot at P, Q, R; and those at (Y′, Z), (Z′, X), (X′, Y) interseet it P′, Q′, R′,; prove that the six points P, Q, R, P′, Q′, R′, all lie on a circle.

1044. (V Ramaswami Aiyar, M.A.) :—Given three straight lines as well as their orthopoles determine the circumcircle as well as the nine points circle of the triangle of reference. Show also how the triangle of reference could be determined.

Solution aud remarks by M. Bhimasena Rao.

It is known that the orthopole of a line given in direction lies on the pedal line of the triangle of reference, which is perpendicular to the given direction and that the distance of the orthopole from the line is constant as the line varies (*Vide*: Orthopole Theorems, Gallatly's *Modern Geometry*), From the given orthopoles draw perpendiculars to the given lines forming a triangle ABC and let the distances of the orthopoles from their correspoding lines be p, q, r. On BC, take $BX = CX' = p$ and draw lines through X and X′ at right angles to BC. Similarly cut off lengths on CA, AB, equal to q, r respectively; and erect perpendiculars to these sides at the points of section. Let the perpendiculars to the sides of ABC, intersect in the points P, Q, R, P′, Q′, R′ as in the figure.



It is now evident that C is the orthopole of X′ R′ and also that of YR′. Therefore the point R′ lies on the circumcircle of the triangle of reference, which for a similar reason passes through P′ and Q′. Since BC is the pedal line through the point C and perpendicular to X′ R′, its pole must line on YR′, the orthopole of which is also C. Considering the point B, it follows by similar reasoning that the pole of BC lies on Z′Q′. Therefore P, the point of intersection of YR′ and Z′Q′ is the pole of BC, and lies on the circumcircle of the triangle of reference' Hence the six points P, Q, R, P′, Q′, R′ are concyclic. This is the result of Question 1043.

For an independant proof, we observe that the projections of PP′ on CA, AB being equal to CA, AB respectively, PP′ must be equal and parallel to the circum-diameter of ABC through A. It follows that PP′=QQ′=RR′. QR′ is parallel to Q′R and QQ′=RR′. Therefore the points QQ′ RR′ are concyclic. RR′ makes with BC the angle 90—A, the angle RR′X=180—A, angle PR′X=C. Hence the angle RR′P=B. Angle RQ′P=180—B. Therefore P lies on the circle QQ. RR′. Similarly P′ lies on this circle.

Since ABC is the triangle formed by the pedal lines of P, Q, R. the reflexion of the orthocentre of PQR in the circumcentre of ABC is the orthocentre of the triangle of reference. (*Vide* : Q. 291, page 128, vol. III, J. I. M. S.)

The circumcircle of the triangle of reference and its ortho-centre being known, the nine points circle is known. The sides of the triangle of reference are the common tangents of the parabolas with foci at P, Q, R and vertex tangents BC, CA, AB.

[Mr. S. Narayanan's question referred to on page 128, vol. III, is "If two triangles inscribed in a circle be such that the pedal lines of the vertices of one with respect to the other are concurrent, then the pedal lines of the vertices of the latter with respect to the former triangle pass through the same point of concurrence". This was published in the December 1909 issue of the J.I.M.S and is quoted by Mr. W. Gallatly on page 50 of his *Modern Geometry*, 2nd Edition. But this property has been stated by Mr. C. E. Youngman in the columns of the *Mathematical Gazette* and a solution of the same by Mr. W. S. Cooney will be found on page 301, vol. April 96 of that Journal. Ed]

Question 442.

(K. J. Sanjana, M. A.); —Obtain a general solution of the Diophantine equation

$$\frac{Ka^2x^2+d}{Kb^2x+d} = \text{square.}$$

$$\frac{75n^2-125}{27n-125} = \text{square.}$$

Find four solutions in rational numbers of the equation.

Solution by N. B. Mitra.

Let $\dfrac{a^2x^2+d}{b^2x+d} = y^2$, ... (1)

and $kx(a^2x^2-b^2y^2) = (y^2-1)$.

If $kx=pd$, then writing n for $\dfrac{ad}{k}$, we have (2)

$$p(n^2p^2-b^2y^2) = y^2-1 \quad \text{or} \quad y^2 = \frac{p^2n^2+1}{pb^2+1}.$$

Assume $y = \dfrac{np}{b}+q$. (3)

$$\therefore \left(\frac{n^2p^2}{b^2}+\frac{2npq}{b}+q^2\right)(pb^2+1) = p^2n^2+1,$$

or $$p^2\left(\frac{n^2}{b^2}+2nbq\right)+p\left(b^2q^2+\frac{2nq}{b}\right)+q^2-1=0.$$

$$i.e.\ p = \frac{\left(-b^2q^2+\dfrac{2nq}{b}\right) \pm \sqrt{\left\{b^4q^4-4nbq^3+8nbq+\dfrac{4n^2}{b^2}\right\}}}{2\left(\dfrac{n^2}{b^2}+2n\ bq\right)}. \quad \ldots \quad (4)$$

Giving any value to q, we get the corresponding value of p then (2) and (3) give the values of x and y: so that, (2), (3) and (4) constitute a general solution.

To get particular *rational* solutions, we proceed thus:—

In order that p may be rational, we must have

$b^4q^4-4nbq^3+8nbq+\dfrac{4n^2}{b^2}$ or $\left(b^2q^2+\dfrac{2n}{b}\right)^2-4bn(q^3+q^2-2q)$ a perfect square.

Put $q^3+q^2-2q=0$. Then $q=0$, 1 or -2.

Also in this case (4) becomes

$$p = \frac{-\left(b^2q^2+\dfrac{2nq}{b}\right) \pm \left(b^2q^2+\dfrac{2n}{b}\right)}{2\left(\dfrac{n^2}{b^2}+2nbq\right)}$$

$$= \frac{(1-q)b}{n+2b^2q} \text{ or } - \frac{q\{b^2y^2+n(b+1)\}}{n(n+2b^2q)}.$$

When $q=0$, $p=\frac{b}{n}$ | or $-\frac{b}{n}$

$\therefore \quad x = \frac{bd}{nk} = \frac{b}{a}$ | or $-\frac{b}{a}$

and $y=1$ | or -1 } (a)

When $q=1$, $p=0$ | $-\frac{b(b^3+2n)}{n(n+2b^3)}$

$x=0$ or | $-\frac{b}{a}$ $\frac{kb+2ad}{2kb^3+ad}$

and $y=1$ or | $\frac{b^3k-ad}{2b^3k+ad}$ } (b)

When $q=-2, p=\frac{3bk}{ad-4b^3k}$ or | $\frac{b}{n}$

$x=\frac{3bd}{ad-4b^3k}$ | $\frac{b}{a}$ } ...(a)

and $y=\frac{ad+8b^3k}{ad-4b^3k}$ | -1

From (*a*) (*b*) & (*c*) we therefore get particular solutions of (1) in rational numbers.

Again if $x=(b)$, $y=f(b)$ be a solution it is evident that $x=(b)$, $y=-f(b)$ and $x=(b)$, $y=(-)f(b)$ are two other solutions. Thus from (*a*) (*b*) & (*c*) we get the following seven different solutions.

	I	II	III	IV	V	VI	VII
x	$\frac{b}{a}$	$-\frac{b}{a}$	0	$-\frac{b}{a}\ \frac{kb^3+2ad}{2kb^3+ad}$	$\frac{b}{a}\ \frac{2ad-b^3k}{ad-2b^3k}$	$\frac{3bd}{ad-4b^3k}$	$\frac{-3bd}{ad+4b^3k}$
y	± 1	± 1	± 1	$\pm\frac{bk^3-ad}{2b^3k+ad}$	$\pm\frac{b^3k+ad}{ab-2b^3k}$	$\pm\frac{ad+8b^3k}{ad-4b^3k}$	$\pm\frac{8b^3k-ad}{ad+4b^3k}$

Again assuming $y=pq'+1$ instead of to $p\frac{n}{b}+q$ and proceeding in the same way, we shall get $q=-\frac{2n}{b}$, $p=\frac{b^3-4n}{3b^2n}$.

Hence we get two other particular solutions :—

x	$\frac{b^3k-4ad}{3\ ab^2k}$	$-\frac{b^3k+4ad}{3ab^2k}$
y	$\pm\frac{b^3k+8ad}{3b^3k}$	$\pm\frac{8\ ad-b^3k}{3b^3k}$

In the nnmerical example given $a=5$, $b=3$, $k=3$, $d=-125$. Hence we get the following nine rational solutions.

x	$\frac{3}{5}$	$-\frac{3}{5}$	0	$-\frac{3507}{2315}$	$\frac{3993}{3935}$	$\frac{1125}{949}$	$-\frac{1125}{301}$	$\frac{2581}{405}$	$\frac{2419}{405}$
y	±1	±1	±1	$\pm\frac{706}{463}$	$\pm\frac{544}{187}$	$\pm\frac{23}{949}$	$\pm\frac{1273}{301}$	$\pm\frac{4919}{243}$	$\pm\frac{508}{243}$

———

Question 624.

(A. N. Raghavachar):—Two parabolas of latera recta $4a$, $4b$, have a common vertex and their axes are inclined at an angle α. If they again intersect at an angle $(\alpha-\beta)$ and if the distance between their foci is c, prove that

$$\frac{c^2}{ab\cos\alpha}=\frac{9t^3s\ (3ts^2-4t+8s)}{4\ (t-2s)^3\ (2t-s)},$$

where t and s stand for tan α and tan β.

Solution by N. Sankara Aiyar.

Let the two parabolas be $y^2=4\ ax$ and $(y\ \cos\alpha+x\ \sin\alpha)^2=4b\ (x\cos\alpha-y\sin\alpha)$.

Let them meet again at $x=\frac{4a}{m^2}$, $y=\frac{a}{m}$.

Then $a\ (m\cos\alpha+\sin\alpha)^2=bm^2\ (\cos\alpha-m\sin\alpha)$ (1)

If θ_1, θ_2 be the angles which the tangents to these curves make with the axis of x, then

$$\tan\theta_1 = \frac{m}{2} \text{ and } \tan\theta_2 = \frac{m\,(1+\sin^2\alpha)-\sin\,\alpha\cos\,\alpha}{1+\cos^2\alpha-m\,\sin\,\alpha\,\cos\,\alpha}$$

$$= \frac{m\,(1+2\,t^2)-t}{2+t^2-mt}$$

from the equation of the second curve as is easily seen, since the equation to the tangent to that curve is

$$(y\cos\,\alpha+x\,\sin\,\alpha)\left(\frac{4a}{m}\cos\,\alpha+\frac{4a}{m^2}\sin\alpha\right)$$

$$=2b\,\cos\,\alpha\left(x+\frac{4a}{m^2}\right)-2b\,\sin\,\alpha\left(y-\frac{4a}{m}\right)$$

and $\alpha-\beta=\theta_1-\theta_2$.

$$\therefore\quad \tan\,(\alpha-\beta)=\frac{2m+mt^2-m^2t-2m\,(1+2t^2)+2t}{4+2t^2-2mt+m^2+2m^2t^2-tm}$$

$$=\frac{2t-tm^2-3mt^2}{4+m^2-3mt+2t^2(1+m^2)}$$

$$\therefore\quad \tan\,\beta = \tan\,(\alpha-\theta_1-\theta_2)$$

$$=\frac{4t+tm^2-3mt^2+2t^3\,(1+m^2)-2t+tm^2+3mt^2}{4+m^2-3mt+2t^2\,(1+m^2)+2t^2-t^2m^2-3mt^3}$$

$$=\frac{2t+2tm^2+2t^3\,(1+m^2)}{4+4t^2-3mt-3mt^3+m^2(1+t^2)}$$

$$=\frac{2t\,(1+m^2).}{4-3mt+m^2}=\beta\ \text{s}$$

$$\therefore\ t-2s=t\,\frac{4-3mt+m^2-4\,(1+m^2)}{4+m^2-3mt}\ldots\ldots \qquad (2)$$

$$=\frac{-3mt\,(m+t)}{4+m^2-3mt}$$

$$2t-s=\frac{2t\times3\times(1-mt)}{4+m^2-3mt}=\frac{6t\,(1-mt)}{4+m^2-3mt}\ldots\ldots \qquad (3)$$

$$3\,ts^2-4\,t+8\,s=3\,ts^2+4\,(2\,s-t)$$

$$=3\,ts^2+\frac{12\,mt\,(t+m)}{4+m^2-3\,mt}$$

$$=12\,t\left\{\frac{t^2\,(1+m^2)^2+m\,(t+m)\,(4+m^2-3mt}{(4+m^2-3\,mt)^2}\right\}$$

$$=12\,t\frac{t^2\,(m^4-m^2+1,\,+2\,tm\,(2-m^2)+m^2\,(4+m^2)}{(4+m^2-3\,mt)^2} \qquad (4)$$

$$\therefore \quad \frac{9\,t^3 s\,(3\,ts^2-4\,t+8\,s)}{4\,(t-2\,s)^2\,(2\,t-s)}$$

$$=\frac{t^2\,(1+m^2)\,\{\,t^2\,(m^4-m^2+1)+2\,tm\,(2-m^2)+m^2\,(4+m^2)\,\}}{m^2\,(t+m)^2\,(1-mt)},$$

$$=\frac{t^2\,\{\,t^2\,(1+m^6)+2\,tm\,(2+m^2-m^4)+m^2\,(4+5\,m^2+m^4)\,\}}{m^2\,(t+m)^2\,(1-mt)}$$

from (2), (3) & (4).

Again $\dfrac{c^2}{ab\cos\alpha}=\dfrac{a^2+b^2-2\,ab\cos\alpha}{ab\cos\alpha}$;

but from (1) $a\cos\alpha=\dfrac{bm^2(1-mt)}{(t+m)^2}$.

$$\therefore \quad a^2=\frac{(1+t^2)\,b^2\,m^4\,(1-mt)}{(t+m)^4}$$

and $abcos\alpha = \dfrac{b^2\,m^2\,(1-mt)}{(t+m)^2}$.

$$\therefore \frac{c^2}{ab\cos\alpha}=\frac{(1+t^2)\,b^2\,m^4\,(1-mt)^2+(t+m)^4\,b^2-2\,b^2\,m^2\,(1-mt)\,(t+m)^2}{b^2\,m^2\,(t+m)^2\,(1-mt)}$$

$$=\frac{m^4\,(1+t^2)\,(1-mt)^2+(t+m)^4-2\,m^2(1-mt)\,(t+m)^2}{m^2\,(t+m)^2(1-mt)}$$

$$=\frac{\{\,(m+t)^2-m^2\,(1-mt)\,\}^2+m^4t^2\,(1-mt)^2}{m^2\,(t+m)^2\,(1-mt)}$$

$$=\frac{t^2}{m^2}\left\{\frac{(t+2\,m+m^2)^2+m^4\,(1-mt)^2}{(t+m)^2\,(1-mt)}\right\},$$

which reduces to

$$\frac{t^2}{m^2}\,\frac{t^2\,(1+m^6)+2\,tm\,(2+m^2-m^4)+m^2\,(4+5m^2+m^4)}{(t+m)^2\,(1-mt)}$$

Hence the result.

Question 821.

(M. K. KEWALRAMANI) :—If through A, B, C lines AXY, BYZ, CZX, are drawn so as to make the same angle θ with AB, BC, CA, respectively, and form the triangle XYZ, prove that $\rho=2\sigma\sin(\omega-\theta)$, where ω is the Brocard angle of the triangle ABC, σ is the radius of the First Lemoine Circle of ABC, and ρ is the radius of the cosine circle of the triangle XYZ.

Solution by Martyn M. Thomas M.A., K. B. Madhava, L. N. Subramaniam and others.

Triangles XYZ and ABC are equiangular, and hence similar. They are also similarly situated with respect to the point S, which is

the Brocard point of the triangle ABC, as well as of XYZ, since ABSX is a cyclic quadrilateral.

$\therefore$ The ratio of similarity, $K=\frac{SX}{SA}=\frac{\sin(\omega-\theta)}{\sin\omega}$.

The sides of the triangle XYZ are Ka, Kb, Kc.

$\therefore$ ρ the cosine radius of $XYZ=\frac{Ka.\,Kb.\,Kc}{K^2a^2+K^2b^2+K^2c^2}=K\frac{abc}{a^2+b^2+c^2}$.

Also $$4\sigma^2=R^2+\left(\frac{abc}{a^2+b^2+c^2}\right)^2$$

$$\therefore\quad \frac{\rho^2}{4\sigma^2}=\frac{K^2}{\left\{\frac{R(a^2+b^2+c^2)}{abc}\right\}^2+1}=\frac{K^2}{\left(\frac{a^2+b^2+c^2}{4\Delta}\right)^2+1}$$

$$=\frac{K^2}{\cot^2\omega+1}=(K\sin\omega)^2$$

$$\therefore\quad \frac{\rho}{2\sigma}=\sin(\omega-\theta).$$

Question 854.

(Martyn M. Thomas M.A.) :—If ρ and σ be the radii of curvature at corresponding points of a curve, and its α—evolute of the β—evolute where α, β are constants, show that

$$\frac{\sigma}{\rho}=\left(\frac{d\rho}{ds}\cos\alpha+\sin\alpha\right)\left(\frac{d\rho}{ds}\cos\beta+\sin\beta\right)+\rho\frac{d^2\rho}{ds^2}\cos\alpha\cos\beta.$$

Solutions by S. Muthukrishnan and several others.

With the usual notation let PQ, QO be neighbouring normals & $P\beta$ & $Q\beta$ two tangents to the β evolute. Then P,Q,O, and β all lie on a circle of which ρ is the diameter in the limit.

Hence $Q\beta=\rho\sin\left(\frac{\pi}{2}-\overline{\beta-d\chi}\right)=\rho\cos(\beta-d\chi)$

where $d\chi$ is the angle between PO & QO

$$=\rho\cos\beta\cos d\chi+\rho\sin\beta\sin d\chi$$
$$=\rho\cos\beta+\rho\sin\beta\,d\chi \text{ in the limit}$$
$$=\rho\cos\beta+ds\sin\beta, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

& if ρ' be the radius of curvature at the corresponding *point* of the β-evolute, we get $\frac{\rho'}{\rho}=\frac{d(Q\beta)}{d\chi}\cdot\frac{d\chi}{ds}$

or $$\frac{\rho'}{\rho}=\frac{d\rho}{ds}\cos\beta+\sin\beta \quad \ldots \quad \ldots \quad (2)$$

Similarly σ can be got from ρ' considering the β evolute as the original curve

$$\frac{\sigma}{\rho'} = \frac{d\rho'}{ds'} \cos\alpha + \sin\alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$= \cos\alpha \cdot \frac{d\rho'}{ds'} \cdot \frac{ds}{d\chi} \cdot \frac{d\chi}{ds'} + \sin\alpha$$

$$= \cos\alpha \cdot \frac{\rho}{\rho'} \cdot \left\{ \rho \frac{d^2\rho}{ds^2} \cos\beta + \left(\frac{d\rho}{ds}\right)^2 \cos\beta + \frac{d\rho}{ds} \sin\beta \right\} + \sin\alpha ;$$

multiplying 2 and 3 and substituting 2 in the R.H.S. we get

$$\frac{\sigma}{\rho} = \cos\alpha \left\{ \left(\frac{d\rho}{ds}\right)^2 \cos\beta + \frac{d\rho}{ds} \sin\beta \right\} + \sin\alpha \left(\frac{d\rho}{ds} \cos\beta + \sin\beta\right)$$

$$+ \rho \frac{d^2\rho}{ds^2} \cos\alpha \cos\beta$$

$$= \left(\frac{d\rho}{ds} \cos\beta + \sin\beta\right) \left(\frac{dp}{ds} \cos\alpha + \sin\alpha\right) + \rho \frac{d\rho^2}{ds^2} \cos\alpha \cos\beta.$$

Question 859.

(HEMRAJ, M.A.) :—A circle intersects a hyperbola in four points A, B, C, D, one focus being within the circle. Prove that the two common chords AB and CD touch a circle such that the two circles have one limiting point at the focus.

Solution by H. R. Kapadia.

The equation of the hyperbola referred to a focus as origin is $x^2+y^2=(ex-a)^2$.

Let $l_1x+m_1y+1=0$ and $l_2x+m_2y+1=0$ be the equations of the common chords AB and CD.

Now $x^2+y^2-(ex-a)^2+\lambda(l_1x+m_1y+1)(l_2x+m_2y+1)=0$... (1)
will represent a circle, if

$$\left.\begin{aligned} \lambda\, l_1\, l_2 - e^2 &= \lambda\, m_1\, m_2 \\ \text{and} \qquad \lambda\,(l_1\, m_2 + l_2 m_1) &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

By suppressing x^2+y^2 from (1), we get

$$-(ex-a)^2+\lambda\,(l_1\,x+m_1\,y+1)\,(l_2\,x+m_2\,y+1)=0. \quad \ldots \quad (3)$$

From (2) we see, that (3) is a $\odot$ and it touches AB and CD.

Thus (1) and (3) have the origin (focus) as a limiting point.

QUESTIONS FOR SOLUTION.

1071. (T. P. Trivedi M.A., L.L.B.)

If $$S_r = \sum_{p=1}^{p=r+1} p^{r+1-p}\,{}_{n-p+1}C_{r-p+1},$$

prove that $$\sum_{r=1}^{r=n} S_r = 1^{n+1}+2^{n}+3^{n-1}+\ldots(n+1)$$

1072. (T. P. Trivedi M. A,, L. L. B.) :— Prove that

$$\int_0^{\pi/2} \sqrt{\tan x}\,\log(\tan x)\,dx = \frac{\pi^2}{\sqrt{2}}$$

$$\int_0^{\pi/2} \sqrt[5]{\tan x}\,\log(\tan x)\,dx = \frac{\pi^2}{4}\left(\frac{3}{\sqrt{5}}-1\right).$$

1073. (N. P. Pandya) :—Give a construction for drawing a common normal to a parabola and a coplanar ellipse.

1074. (V. Tiruvenkatachari) :—If $a_1, a_2, a_3, \ldots a_n$ be n unequal positive numbers, show that

(i) $L_n = (G_1\, G_3\, G_5\ldots)/(G_2\, G_4\ldots)$,

(ii) $G_n = (L_1\, L_3\, L_5\ldots)/(L_2\, L_4\ldots)$,

where G_r denotes the product of the G. C. M's of the numbers r at a time and L_r denotes the similar product of the L. C. M's.

1075. (M.K. Kewalramani):—A B C is a triangle and P is its orthocentre. P A, P B, P C are produced to A^1, B^1, C^1, respectively such that $A\,A^1 = r_1$, $B\,B^1 = r_2$, $C\,C^1 = r_3$ where r_1, r_2, r_3 have their usual meanings If ξ, η, ζ are the normal coordinates of the incentre of the triangle A B C with respect to triangle $A^1\,B^1\,C^1$ show that

$$\frac{(S\cos\frac{B}{2}-\eta)}{\operatorname{cosec}\frac{B}{2}} = \frac{(S\cos\frac{A}{2}-\xi)}{\operatorname{cosec}\frac{A}{2}} = \frac{(S\cos\frac{C}{2}-\zeta)}{\operatorname{cosec}\frac{C}{2}} = \frac{r}{2}$$

1076. (S. Ramanujam F. R. S.) show that

(1) $\sqrt[6]{7\sqrt[3]{20}-19} = \sqrt[3]{5/3} - \sqrt[3]{2/3}$

(2) $\sqrt[6]{4\sqrt[3]{2/3}-5\sqrt[3]{1/3}} = \sqrt[3]{4/9} - \sqrt[3]{2/9} + \sqrt[3]{1/9}$

1077. (SADANAND):—Prove that the result of eliminating α, β, γ and δ from the following equations

$$\begin{vmatrix} 0 & a_1 & a_5 & \alpha & 1 \\ a_1 & 0 & a_2 & \beta & 1 \\ a_5 & a_2 & 0 & \gamma & 1 \\ \alpha & \beta & \gamma & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} 0 & a_2 & a_6 & \beta & 1 \\ a_2 & 0 & a_3 & \gamma & 1 \\ a_6 & a_3 & 0 & \delta & 1 \\ \beta & \gamma & \delta & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0;$$

$$\begin{vmatrix} 0 & a_3 & a_5 & \gamma & 1 \\ a_3 & 0 & a_4 & \delta & 1 \\ a_5 & a_4 & 0 & \alpha & 1 \\ \gamma & \delta & \alpha & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & a_4 & a_6 & \delta & 1 \\ a_4 & 0 & a_1 & \alpha & 1 \\ a_6 & a_1 & 0 & \beta & 1 \\ \delta & \alpha & \beta & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0;$$

is the equation $\begin{vmatrix} 0 & a_1 & a_5 & a_4 & 1 \\ a_1 & 0 & a_2 & a_6 & 1 \\ a_5 & a_2 & 0 & a_3 & 1 \\ a_4 & a_6 & a_3 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0.$

1078. (A. C. L. WILKINSON):—Show that the nine vertices of the three triangles formed by the common tangents of three parabolas taken in pairs lie on a conic.

1079. (A. C. L. WILKINSON):—Let ABCD be any four coplanar points (non cyclic). Let α be the isogonal conjugate of A with respect to the triangle BCD, etc. Denote the common pedal circle of A, α with respect to BCD by (A), and let O be the common point of intersection of (A), (B), (C), (D). Prove that:—

(1) the six joins of the four points A, B, C, D, bisect at right angles the six joins of the four points α, β, γ, δ, *viz*, AB bisects $\gamma\delta$ at right angles and that the meet of AB, $\gamma\delta$ is the point of intersection of the circles (C), (D) other than O.

(2) that the seven points of intersection of the four circles (A) (B), (C), (D) lie on a conic whose axes are parallel to the asymptotes of the rectangular hyperbola through A, B, C, D and which also passes through the centre of the rectangular hyperbola through α, β, γ, δ.

(3) That this conic will be a rectangular hyperbola if two of the sides of the quadrilateral ABCD are parallel.

(4) that the sum of the reciprocals of the powers of A, B, C, D with respect to (A), (B), (C), (D) respectively is zero.

Mathematics from Periodicals.

Proceedings of the Royal Society, Series A, Vol. 95, No. A 670.

Vacuum Arc Spectra of various Elements in the Extreme Ultra-violet, by J. C. Mclennan, F.R.S., D.S. Ainslie, and D. S. Fuller. (For tin, lead, and thallium, new lines have been observed and measured below λ 1700 A and for iron, cobalt, nickel and carbon, several new lines have been photographed in the region between λ1850A and λ1400A. Other measurements also are given covering and confirming previous work.)——*An Investigation of the Ionising Power of the Positive Ions from a glowing Tantalum Filament in Helium*, by Frank Horton Sc. D Ann Catherine Davies, D. Sc. (As the result of experiments involving the ionisation of helium gas by collisions, the authors were led to the conclusion that the minimum ionising potential for positive ions was very much greater than that for electrons.)——*Wave Resistance: Some Cases of Three-dimensional Fluid Motion*, by T. H. Havelock, F. R. S. (The object of the paper is to calculate the wave resistance for assigned surface pressures of considerable generality, taking the surface pressure to be two dimensional and the wave-pattern like that produced by a ship. Starting with the expressions for the velocity potential of the fluid motion and the surface elevation given in Lamb's *Hydrodynamics*, p. 425, for an initial impulse symmetrical about the origin, the auther limits his discussion to the case where the system has been in motion for a long time. He then analyses the surface elevation into plane wave constituents and obtains the wave-resistance by considering first a simple wave constituent and then summing up for all of them. A few simple cases are worked out for illustration, such as that of a prolate spheroid moving in an infinite liquid with a given velocity in the direction of its axis of symmetry)——*Experiments Demonstrating an Electrical Effect in Vibrating Metals*, by Admiral Sir H. B. Jackson and G. B. Bryan, D. Sc. (The paper describes certain experiments, indicating that part of the energy of a vibrating wire gives rise to an electro-motive force in it producing a sufficiently strong alternating current).

The Philosophical Magazine and Journal of Science, Vol. 37, No. 220. April 1919.

On the Problem of Random Vibrations and of Random Flights in One, Two or Three Dimensions, by Lord Rayleigh, O.M., F.R.S. (When a number (n) of isoperiodic vibrations of amplitude (l) are combined, and the spaces are taken at random, the probability that

the resultant amplitude lies within the limits r and $(r+dr)$ when n is very great was given by the author (*Phil. Mag.*, Vol. X, p. 73, 1880) to be $(2/nl^2)\,\mathrm{Exp}\,(-r^2/nl^2)rdr$. The object of the present paper is to consider the question further with extensions to three dimensions and with a comparison of results for one, two and three dimensions. The last case has no applications to random vibrations; it applies to random flights only. The expressions found for the probability for one and three dimensions are found to approximate respectively to $\sqrt{\left(\frac{1}{2n\pi}\right)}e^{-r^2/2nl^2}dr/l$ and $\frac{3\sqrt{6}.\,r^2}{\sqrt{\pi}.\,n^{3/2}}\,e^{-3r^2/2nl^2}\,dr/l^3$ — *On the Fundamental Laws of Electrical Action*, by Magh Nad Saha, M.Sc. (An attempt is made to determine the law of attraction between two moving electrons, with the aid of the New Electro-dynamics as modified by the principle of relativity. With the help of Minkowskis' four-dimensional analysis, the author recasts the results of Lienard and Wiechert on the field produced by a moving electron and deduces a Lagrangian function controlling the motion of two electrons round each other which for small velocities is identical with that tentatively assumed by Clausius). *Experiments on the High-Tension Magneto*, by Norman Campbell, Sc.D. *On the Dispersion of Diamond*, by L. Silberstein, Ph.D. (The object is to apply the principle of inter-action of atoms, to the refractive properties of diamond, considered as a known assemblage of fixed "atomic centres" each considered a single dispersive electron and becoming a doublet in presence of an external electric field. The dispersion formula of diamond, so deduced, is $\mu^2=1+\frac{a}{u_o-u}$, where $u=\frac{1}{\lambda^2}$ and $u_o=\frac{1}{\lambda_o^2}$ —λ and λ_o being the incident (in vacuo) and free wave lengths respectively, belonging to the electron of the carbon atom— and $a=n\frac{e^2/m}{4\pi^2c^2}$, n being the number of carbon-atoms per unit volume of the crystal, c the velocity of light in vacuo, and e and m the charge and mass respectively of a single electron; the assumptions made in obtaining the formula fail to hold good only in the case of very long infra-red or too short waves. A comparison with actual observations shows that the formula is fairly accurate.—*Examples of Operational Methods in Mathematical Physics*, by T. J. I'A, Bromwich, Sc.D. (The author works out a number of examples in Conduction of Heat and in Electricity to illustrate the merits of operational method, advocated by Heaviside and develops a more elementary treatment of the same. His new formula is stated thus:—Suppose we have found by operational methods the symbolical equation $x=\frac{F(p)}{\Delta(p)}(G\,t)$, where

p stands for the operator d/dt and G is a constant; then the solution is to be interpreted as $x = G\left\{ N_o t + N_1 + \sum \frac{F(\alpha)}{\alpha^2 \triangle'(\alpha)} e^{dt} \right\}$, where $p = \alpha$ is any root of $\triangle(p) = o$, the summation extends to all such roots and N_o, N_1 are defined by the algebraic expression $F(p)/\triangle(p) = N_o + N_1 p + N_2 p^2 + \ldots\ldots$, the solution having the property of reducing to zero at $t = o$. The formula is applied to obtain expression for the mean temperature of a sphere and a cylinder and allied problems in Conduction of Heat and also to problems of Induction Balances).—*On the Origin of Spectra and Planck's Law*, by Sir J. J. Thomson. (The paper discusses the question whether, if we regard the electrons and positive charges in the atoms of the various elements as centres of force varying inversely as the square of the distance, we have the potentiality of explaining by mechanical principles, the properties of the atom. The explanation of some of the properties–such as, the specific inductive capacity of the gas, the formation of molecules by union with other atoms etc.—seems to be within the scope of this simple system; but other properties, such as the emission of different spectra by gases, are not amenable to a similar treatment. The author supposes the field of force round the positive charge, to vary inversely as the square of the distance at large distances from the atom and, in the atom itself, to change backwards and forwards between attraction and repulsion as it would do if the expression for the force contained the factor $\sin(c/r)/(c/r)$. He also imagines that in addition to the electric field inside the atom, there is a magnetic one and that the frequencies of the vibrations of the electrons are determined by the magnetic and not electric forces. It is then shown that if we assume a simple relation between the electric and magnetic forces, we are led to Planck's Law and a physical concept of the Quantum Theory).—*A General Formula for the Moments of the Normal Correlation Function of any Number of Variables*, by S. D. Wicksell, (A formula is obtained for evaluating

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots\ldots \int_{-\infty}^{\infty} dx_n x_1^{k1} x_2^{k2} \ldots\ldots x_n^{kn} \phi(x_1, x_2, \ldots\ldots x_n),$$

where ϕ is the normal correlation function of $x_1, x_2, \ldots\ldots x_n$, the variates reckoned from their respective means and expressed in their respective dispersions as units).—*Mechanical "Resonators" under Double Forcing*, by E. H. Barton, F. R. S.—*Resonance Radiation of Sodium Vapour excited by one of the D. lines*, by R. W. Wood and F. L. Mohler.

Philosophical Transactions of the Royal Society of London. Series A, Vol. 218.

Bakerian Lecture, 1917; *the Configurations of Rotating Compressible Masses* by J. H. Jeans, M. A; F. R. S. (A mass of incompressible matter shrinking while rotating assumes first the shape of a spheroid, then that of an ellipsoid, afterwards becoming unstable and probably dividing into two detached masses after passing through a series of pear-shaped configurations. In the present investigation, it is found that, except for an alternative possibility to be discussed later, a shrinking compressible mass will experience a very similar sequence of changes. It will pass through a series of shapes which from their similarity to the spheroids and ellipsoids of incompressible masses, may be described as pseudo-spheroids and pseudo-ellipsoids. The pseudo-spheroids become unstable when a certain degree of flatness is reached and give place to a series of pseudo-ellipsoids, these in turn become unstable when a certain degree of elongation is reached and give place to a series of pear-shaped figures which probably end by fusion into detached masses. The method adopted consists in making use of the known solution for an incompressible mass as generating solution and then obtaining by a method of successive approximation an expansion of the potential function in powers of ε where $\varepsilon=(\rho_o-\sigma)/\rho_o\rho_o$ being the density at the centre and σ the density at the boundary of the mass).

Transactions of the American Mathematical Society, Vol. 19, No. 2, April 1918—

Proof that certain Ideals in a Cyclotomic Realm are Principal Ideals, by H. H. Mitchell (The author obtains the following theorem: Let λ denote any prime such that $\lambda-1$ is not a power of 2, and let $2e$ denote a divisor of $\lambda-1$ such that $(\lambda-1)/(2e)$ is an odd integer other than unity. Let q be a prime such that $q^{(\lambda-1)/2e}\equiv 1$, mod λ. Then in the realm of degree $2e$ determined by the $2e$ periods formed from λth roots of unity, it is known that q is the product of $2e$ conjugate prime ideals, which form e conjugate imaginary pairs. Further, let h denote the "first factor" of the class-number of the realm. Then, if one ideal be selected from each of the e conjugate imaginary pairs, and the product of these be raised to the power h, the result is a principal ideal).—*The Order of Primitive Groups (III)* by W. A. Manning—(The principal theorem proved is the following:—Let q be any positive integer greater than 5, and let p be any prime greater than $2q-2$; then the degree of any primitive group G that contains a substitution of order p and degree qp, but none of order p and of degree less than qp, does not exceed $qp+4q-4$; if G is not triply or (doubly) transitive its degree does not exceed $qp+4q-6$ or $(qp+4q-7)$; the order of G is not

THE JOURNAL

OF THE

Indian Mathematical Society

DECEMBER 1919 [No. 6.

The late Mr. S. Ramanujan, B.A., F.R.S.

We deeply regret to announce the untimely death of Mr. S. Ramanujan, B.A., F.R.S., on Monday, the 26th April 1920, at his residence at Chetput, Madras. An account of his life and work will appear in a subsequent issue of this Journal.

It has the honour to invite for participation in the work of the Congress the Mathematicians of Allied Countries and of those Neutral Countries a list of which was drawn up by the third Interallied Conference of Academies held at Brussels in July 1919. In order to facilitate making proper arrangements the Committee will be greatly obliged to all such persons as will intimate their acceptance as early as possible.

The Congress is to be opened on 22nd September 1920.

It will be divided into four sections which shall be sub-divided into as many sub-sections as the number and nature of the communications submitted may require.

Section I. Arithmetic—Algebra—Analysis.

II. Geometry.

III. Mechanics—Mathematical Physics—Applied Mathematics.

IV. Questions—Philosophical, Historical and Pedagogical.

Official accounts of the work of the Congress, at least in the form of a summary, will be sent to each of the subscribers.

A detailed programme giving information regarding journey and accommodation as well as proposed receptions and excursions will be published subsequently.

(Sd.) E. PICARD,
The President of the French National Committee.

Additional Information.

I. The subscription for Membership to the Congress is 60 francs (about Rs. 15) per head, payable to Mr. Valiron, Treasurer of the Congress, (52, Allee de la Robertson, Strassburg).

A contribution of 30 francs will entitle any person of the family of a member to the same privileges, with the exception of the despatch of a copy of the Official Report.

II. All those that desire to present to the Congress one or more papers are requested to notify the same to Mr. Koenigs, General Secretary of the French National Committee (96, Boulevard Raspail, Paris) and to inform him the subject matter before 1st July 1920.

III. Further information may be obtained from either Mr. Koenigs or Mr. Villat, President of the Local Committee of Organisation of the Congress (11, Rue du Mareckal Letain, Strassburg), or Mr. Golbrun, Secretary of the French National Committee (14, Avenue Emile-Deschanel, Paris.)

Computation of the Ephemeris of a Planet and a Comet.

By R. N. Apte, M.A., LL.B., F.R.A.S.

§ 1. An outline of the method of computing an ephemeris is indicated in Text-books like Tisserand's *Orbites* p. 88 and Picart's *Orbites* p. 43.* But there are several points which require an examination in detail from a computer's point of view. A popular treatment of the subject is given in *Popular Astronomy* (1901) and Crommelin wrote a paper in 1895 on a rough ephemeris of a parabolic orbit and recently (1915) of an elliptic orbit.

An ephemeris gives the moving body's Right Ascension (R. A.) and declination (dec.) and the log. of its distance from the Earth. The unit of distance is the Earth's mean distance from the Sun. The ephemeris is computed from the given elements of an orbit.

§ 2. The elements of an elliptic orbit are :—

1. The epoch.
2. M, the mean anomaly.
3. ω, the angular distance of the Perihelion from the Node.
4. Ω, the longitude of the node.
5. i, the inclination of the orbit to the ecliptic.
6. ϕ, where $\sin \phi = e$, the eccentricity.
7. μ, the mean motion.
8. $\log a$, where a is the major axis

The instant of perihelion passage is determined from the above.

The period $= a^{\frac{3}{2}}$ years.

The elements of a parabolic orbit are :—

1. The time of passing perihelion.
2. ω
3. Ω
4. i

} defined as above.

5. $\log q$, where q is the distance of the perihelion from the Sun.

* Also Plummer's *Dynamical Astronomy* received in India long after this paper was read. The subject is given there in a very condensed form with very few new points.

The symbol Ω denotes the ascending node where the body is crossing the ecliptic from South to North, and $\mho$ denotes the descending node. S being the sun, the straight line Ω S $\mho$ lies in both the plane of the orbit and the plane of the ecliptic.

The longitude Ω is measured eastwards.

ω is called the argument of Perihelion and is the angle Ω S A. It is to be noted that ω is measured in the direction in which the body is moving and therefore is greater than 180° when the body passes the ascending node after it has passed the perihelion. Hence obviously the angle A S $\Omega = 360^0 - \Omega$, where A is the perihelion.

π is the longitude of the perihelion and sometimes this is given instead of ω. π is measured along the ecliptic from Υ (the first point of Aries) to Ω and the along the body's orbit from Ω to A *in the direction of the body's motion.* If therefore $\Omega + \omega > 360^0$, $\pi = \Omega + \omega - 360^0$.

i is the inclination of the orbit to the ecliptic. If the motion is direct $i < 90^0$ and if retrograde $i > 90^0$ and $< 180^0$. If the elements are given (considering i to be the smaller angle between the planes) in the computation when the motion is retrograde, we must put i = supplement of inclination $\pi = 2\,\Omega$ — given longitude of perihelion. In the elliptic orbit $q = a\,(1-e)$.

The epoch is the instant for which the value of M, the mean anamoly, is given. It is better to put M_o for the mean anamoly at the epoch and M for the mean anamoly at the time for which computation is to be made.

μ is the increase of the mean anamoly M in a mean solar day and is usually given in seconds.

Besides the above, we use the following :—

t is the interval of time expressed in solar days (and decimals of solar day) between T, the instant of perihelion passage, and the instant of time for which the computation is to be made; t is *positive* after perihelion passage and *negative* before it.

v is the *true anomaly.* Thus if P is the moving body, $v = \angle$ASP. This is positive after perihelion passage and is regarded as negative before perihelion passage.

r is the *radius vector* of the body, that is, its distance from the sun in terms of the unit mentioned above.

Δ is the distance of the body from the Earth in terms of the same unit.

u, the argument of latitude, is the angle measured in the direction of motion through which P has moved round the sun after passing its ascending node, so that $u = v + \omega$.

§ 3. *Co-ordinates of the body P* :—

1. In the equatorial-heliocentric system, taking as axes of $+x$, $+y$ $+z$, respectively, lines from S to points whose R.A. and decl. are (0°, 0°) (90°, 0°) (0°, 90°) ; let P be (x, y, z) and let the direction cosines of P be (c_1, c_2, c_3).

2. In the ecliptic-heliocentric—Υ system, taking as axes of $+x$, $+y$, $+z$ respectively, lines from S to points whose longitudes and latitudes are (0°, 0°) (90°, 0°) (0°, 90°), Let P in direction cosines be (c_1', c_2', c_3').

3. In the ecliptic-heliocentric—Ω system, taking as axes of $+x$, $+y$, $+z$ respectively, lines from S to points whose angular co-ordinates are (0°, 0°) (90°, 0°) (0°, 90°) angles being measured along and perpendicular to the ecliptic, let P in direction cosines be (c''_1, c''_2, c''_3).

4. In the orbital-heliocentric—Ω system taking as axes of $+x$, $+y$, $+z$) respectively, lines from S to points whose angular co-ordinates along and perpendicular to the orbit are (0°, 0°) (90°, 0°) (0°, 90°), let P in direction cosines be (c_1''', c_2''', c_3''').

Then, as in the author's paper on *Direction Cosines* (J. I. M. S., Vol. I, p. 123).

	c_1''' $=\cos\omega$	c_2 $=\sin\omega$	c_3''' $=0$
c_1''	1	0	0
c_2''	0	$\cos i$	0
c_3''	0	$\sin i$	0

	c_1''	c_2''	c_3''
c_1'	co Ω	$-\sin\Omega$	0
c_2'	$\sin\Omega$	$\cos\Omega$	0
c_3'	0	0	1

	c_1'	c_2'	c_3'
c_1	1	0	0
c_2	0	$\cos\varepsilon$	$-\sin\varepsilon$
c_3	0	$\sin\varepsilon$	$\cos\varepsilon$

$\therefore$

	c_1'''	c_2'''	c_3'''
c_1	p_1	p_2	p_3
c_2	q_1	q_2	q_3
c_3	r_1	r_2	r_3

... ... (1)

where ε = obliquity
and
$p_1 = \cos\Omega$
$p_2 = -\sin\Omega$
$p_3 = 0$
$q_1 = \sin\Omega\cos\varepsilon$
$q_2 = \cos\Omega\cos\varepsilon$
$q_3 = -\sin\varepsilon$
$r_1 = \sin\Omega\sin\varepsilon$
$r_2 = \cos\Omega\sin\varepsilon$
$r_3 = \cos\varepsilon.$

... (2)

[We have obviously

$x = rc_1,\ y = rc_2,\ z = rc_3,$ (3)

$x = r(\cos\overline{v+\omega}\cos\Omega - \sin\overline{v+\omega}\sin\Omega\cos i),$

$y = r\,(\cos\overline{v+\omega}\sin\Omega\cos\varepsilon + \sin\overline{v+\omega}\cos\Omega\cos i\cos\varepsilon - \sin\overline{v+\omega}\sin i\sin\varepsilon)$

$z = r(\cos\overline{v+\omega}\sin\Omega\sin\varepsilon + \sin\overline{v+\omega}\cos\Omega\cos i\sin\varepsilon + \sin\overline{v+\omega}\sin i\cos\varepsilon).$... (4)

These expressions may be simplified. Put

$\sin a\sin A = \cos\Omega$
$\sin a\cos A = -\cos i\sin\Omega$ } (i)

$\sin b\sin B = \sin\Omega\cos\varepsilon$
$\sin b\cos B = \cos\Omega\cos i\cos\varepsilon - \sin i\sin\varepsilon$ } ... (ii)

$\sin c\sin C = \sin\Omega\sin\varepsilon$
$\sin c\cos C = \cos\Omega\cos i\sin\varepsilon + \sin i\cos\varepsilon.$ } ... (iii)

... ... (5)

Then

$$\left.\begin{aligned} x &= r \sin a \sin (\mathrm{A}+v+\omega) \\ y &= r \sin b \sin (\mathrm{B}+v+\omega) \\ z &= r \sin c \sin (\mathrm{C}+v+\omega) \end{aligned}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

which may be written as

$$\left.\begin{aligned} x &= r \sin a \sin (\mathrm{A}+u) \\ y &= r \sin b \sin (\mathrm{B}+u) \\ z &= r \sin c \sin (\mathrm{C}+u) \end{aligned}\right\}; \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

or, again, putting

$$\sin a = k_1, \ \sin b = k_2, \ \sin c = k_3 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

and

$$\mathrm{A}_1 = \mathrm{A}+\omega, \ \mathrm{B}_1 = \mathrm{B}+\omega, \ \mathrm{C}_1 = \mathrm{C}+\omega, \quad \ldots \quad \ldots \quad (9)$$

which substitution is found better for computation, we may have

$$\left.\begin{aligned} x &= k_1 r \sin (\mathrm{A}_1+v) \\ y &= k_2 r \sin (\mathrm{B}_1+v) \\ z &= k_3 r \sin (\mathrm{C}_1+v). \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

k_1, k_2, k_3 are taken to be always positive and hence sin A, cos A must have respectively the signs of cos Ω and $-\sin\Omega \cos i$; and A must be chosen accordingly to lie in the proper quadrant.

If the elements be given with reference to the ecliptic, in equations (1), (2), c_1, c_2, c_3 are not required and we get only c_1', c_2', c_3' in terms of c_1'', c_2'', c_3''. Thus putting

$$x = rc_1', \ y = rc_2', \ z_1 = rc_3',$$

and as before taking

$$\left.\begin{aligned} \sin a \sin \mathrm{A} &= \cos \Omega \\ \sin a \cos \mathrm{A} &= -\cos i \sin \Omega \\ \sin b' \sin \mathrm{B}' &= \sin \Omega \\ \sin b' \cos \mathrm{B}' &= \cos i \cos \Omega \end{aligned}\right\}$$

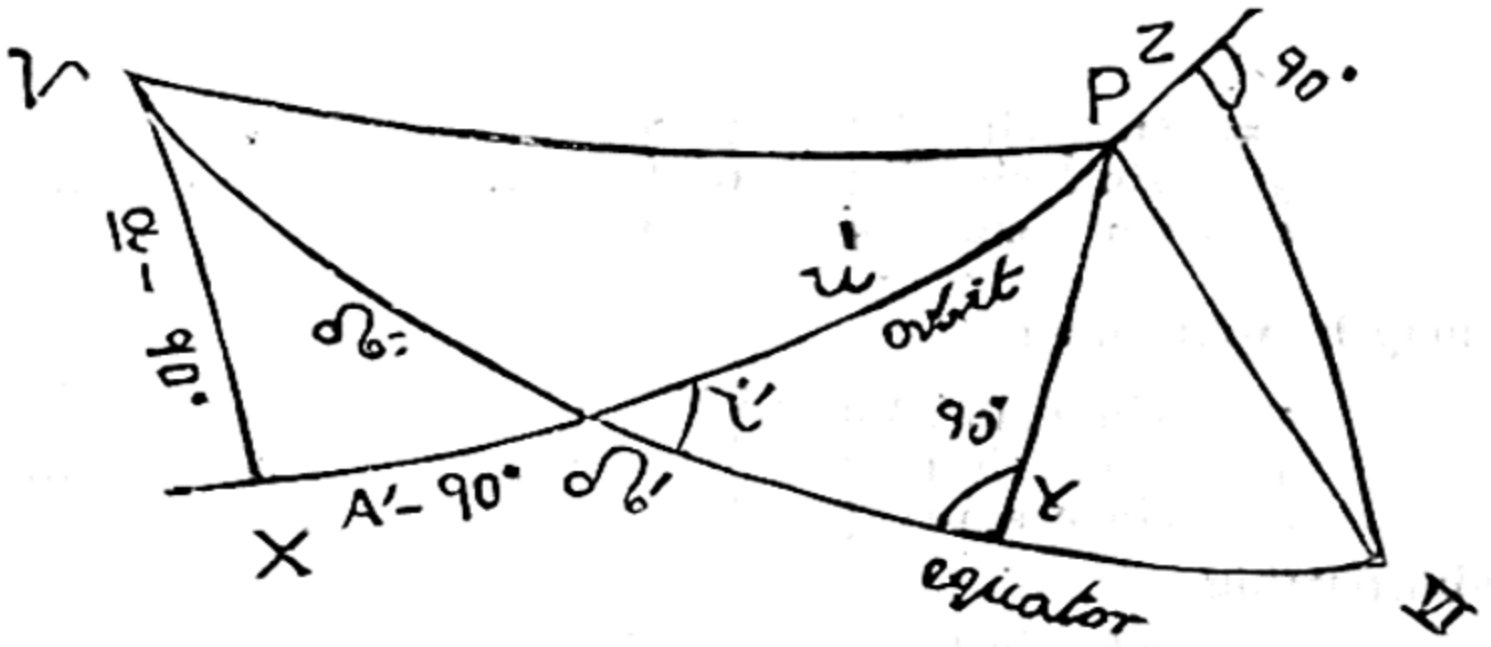
as in (5), obliquity being not required in this case, we have

$$\left.\begin{aligned} x_1 &= r \sin a \sin (\mathrm{A}+\omega+v) \\ y_1 &= r \sin i \sin (\mathrm{B}+\omega+v) \\ z_1 &= r \sin i \sin (\omega+v) \end{aligned}\right\}, \quad \ldots \quad \ldots \quad (6\mathrm{A})$$

which by the substitutions (8), (9) may be written like (10),

$$\left.\begin{aligned} x_1 &= k_1 r \sin (\mathrm{A}_1+v) \\ y_1 &= k_2 r \sin (\mathrm{B}_1'+v) \\ z_1 &= k_3 r \sin (\omega+v). \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (10\mathrm{A})$$

§ 4. The six quantities sin a, sin b, sin c, or k_1, k_2, k_3, and A, B, C are called the *Constants of Gauss*. k_1, k_2, k_3 are always to be taken as positive. These are constant for the same orbit as they depend only on the elements of the orbit and the obliquity of the ecliptic.



§ 5. Taking this figure on the celestial sphere. where ϒX is a great circle perpendicular to the orbit or rather the trace of it on the celestial sphere and VI is a point whose R. A. and dec. are (90°, 0°), we have from triangle ϒ Ω'P

$$\cos \Upsilon P = \cos \Omega' \cos u' - \sin \Omega' \sin u' \cos i' \quad \ldots \quad \ldots \quad \text{(i)}$$

and from triangle ϒXP:

$$\cos \Upsilon P = \sin a \sin (A' + u')$$
$$= \sin a \sin A' \cos u' + \sin a \cos A' \sin u' \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\left.\begin{aligned} \text{and} \quad & \sin a \sin A' = \cos \Omega' \\ & \sin a \cos A' = -\sin \Omega' \cos i' \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

which is obvious from the triangle ϒXΩ';

and $x = r \cos \Upsilon P = r \sin a \sin (A' + u')$.

So from the triangles PZVI, PΩ'Y, we get

$$y = r \sin b \sin (B' + u')$$
$$z = r \sin c \sin (C' + u')$$
$$(c' = i', \ C' = 0).$$

We have thus $A + u = A' + u'$

$$A + \omega + v = A' + \omega' + v$$

and $$A = A' + (\omega' - \omega)$$
$$B = B' + (\omega' - \omega)$$
$$C = C' + (\omega' - \omega)$$

and $\omega' - \omega$ is the arc of the orbit comprised between the equator and the ecliptic.

So we have a geometrical interpretation of the constants of Gauss,

[See : Moulton's *Celestial Mechanics*, p. 189.]

§6. From equations 5 (i)

$$\tan A = \cot \Omega \sec (180 - i) \quad \ldots \quad \ldots \quad (11)$$

and putting

$$\left.\begin{aligned} k \sin N &= \sin i \\ k \cos N &= \cos \Omega \cos i \end{aligned}\right\} \quad \ldots \quad \ldots \quad (12)$$

$$\cot B = \frac{\cos i \cos (N+\varepsilon)}{\tan \Omega \cos N \cos \varepsilon} \quad \ldots \quad \ldots \quad (13)$$

$$\cot C = \frac{\cos i \sin (N+\varepsilon)}{\tan \Omega \cos N \sin \varepsilon} \quad \ldots \quad \ldots \quad (14)$$

$$k_1 = \cos \Omega \operatorname{cosec} A \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$k_2 = \sin \Omega \cos \varepsilon \operatorname{cosec} B \quad \ldots \quad \ldots \quad (16)$$

$$k_3 = \sin \Omega \sin \varepsilon \operatorname{cosec} C', \quad \ldots \quad \ldots \quad (17$$

We have also the check equation

$$\tan i = \frac{k_2 k_3 \sin(C-B)}{k_1 \cos A}. \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

From these equations the constants of Gauss are computed. As they are constant for the orbit, they are computed only once for all places in the orbit.

Example.

$\Omega = 85^\circ\ 57'\ 34''\cdot4$
$i = 11\ \ 42\ \ 8\cdot0$
$\varepsilon = 23\ \ 27\ \ 8\cdot0$
$A = 175\ 52\ 26\cdot7$
$k_1 = 9\cdot990925$
$N = 71^\circ\ \ 12'\ 46''\cdot8$
$k = 9\cdot330899$
$B = 91^\circ\ 5'\ 27''\cdot1$
$k_2 = 9\cdot961553$
$C = 61^\circ\ 43'\ 32''\cdot8$
$k_3 = 9\cdot653962.$

Check $[C - B] = 29^\circ\ 21'\ 54''\cdot3$

§7. In the elliptic orbit

$$M = \mu t = E - e \sin E. \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

The Epoch and the instant of time for which the computation is required should be of the same place and the interval t between them should be found. If they are not of the same place, reduction to some place is necessary before we get t. For instance: if the Epoch is Oct. 31·5 Berlin mean time, and the computation is required at Greenwich Dec. 25 mean midnight, we reduce the former and take the Epoch to be 31·462790 Greenwich mean time.

Hence $t = 55{\cdot}037210$ and we get μt.

M being known, equation (19) gives E. This is a transcendental equation and several methods of solving it have been discovered.

The angles in equation (19) are in circular measure and should be reduced to minutes or seconds. The equation (19) thus becomes

$$M = \mathrm{E} - e'' \sin \mathrm{E}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where $e'' = e$ multiplied by $3437'{\cdot}747$, the number of minutes in one radian, or e multiplied by $206264''{\cdot}8$, the number of seconds in one radian.

For ordinary purposes it is convenient to have the equation in minutes.

Let E_0 be a value of E obtained from the first three terms of the equation

$$E = M + e \sin M + \frac{e^2}{2} \sin 2M + \frac{e^3}{24}(9 \sin 3M - 3 \sin M) + \ldots \quad \ldots \quad (21)$$

and let $E_0 + \triangle E_0$ be the true value of E, then as given in Text-books,

$$\triangle E_0 = \frac{M - M_0}{1 - e \cos E_0} \quad \ldots \quad \ldots \quad (22)$$

Applying this correction to E_0, let the value of E be $E_1 = E_0 + \triangle E_0$; proceeding as before we get

$$\triangle E_1 = \frac{M - M_1}{1 - e \cos E_1} \quad \ldots \quad \ldots \quad (23)$$

Usually two applications of the formula will be found sufficient. Thus, if $M = 34^\circ\ 19'\ 36''{\cdot}14$, $\log e = 9{\cdot}7442503$,

$$\begin{aligned} \mathrm{E}_1 &= 62^\circ\ 28'\ 2''{\cdot}4 \\ \mathrm{E}_2 &= 62\ \ 32\ \ 23{\cdot}70 \\ \mathrm{E}_3 &= 62\ \ 32\ \ 25{\cdot}79. \end{aligned}$$

The calculation of these corrections is greatly facilitated by the Tables in *Bulletin Astronomique*, 1885, or by Banschinger's Tables XI and XII.

If E is very small we can at once assume $\mathrm{E} = \sin \mathrm{E}$ and equation (19) becomes

$$E = \frac{\mu t}{1 - e},$$

or when great accuracy is not required, equation (21) is,

$$E = \frac{\mu t}{1 - e} - \frac{1}{6} \frac{e}{(1 - e)} \cdot \left(\frac{M}{1 - e}\right)^3.$$

Also $$E=\frac{\mu t}{1-e}-\frac{e}{1-e}k\left(\frac{\mu t}{1-e}\right)^3,$$

where k should be chosen so as to secure the best distribution of errors within the desired range of values of E.

Mr. Crommelin finds that $k=[2\cdot1167]$ makes the best distribution up to 21°. $E=\frac{\mu t}{1-e}$ works well for values of E up to 3°. The value of k within the brackets is a logarithm and 2 stands for -8.

§8. After E is fonnd, the next step is to find v and r. We have the usual formulæ :—

$$\left.\begin{aligned}
&r\sin v=b\sin E=a\cos\phi\sin E\\
&r\cos v=a\cos E-ae=a(\cos E-\sin\phi).
\end{aligned}\right\}\ \ldots\quad (i)$$

Also

$$\sin\frac{v}{2}\sqrt{r}=\sin\frac{E}{2}\sqrt{a(1+e)}\qquad\ldots\quad (ii)$$

$$\left.\begin{aligned}
&\cos\frac{v}{2}\sqrt{r}=\cos\frac{E}{2}\sqrt{a(1-e)}\\
&\text{and}\qquad r=a(1-e\cos E)\\
&\tan\frac{E}{2}=\tan\frac{v}{2}\sqrt{\frac{1-e}{1+e}}=\tan\frac{1}{2}v\tan(45^\circ-\frac{1}{2}\phi)\\
&\text{or}\quad \sin\frac{1}{2}(v-E)=\sin\frac{\phi}{2}\sin E\sqrt{\frac{a}{r}}.
\end{aligned}\right\}\ \ldots\quad (25)$$

These are arranged in different groups so as to give different ways of obtaining v and r.

Example.

$M=52^\circ\ 48'\ 12''\cdot9$ $\qquad$ $\phi=3^\circ\ 43'\ 0''\cdot3$, $\log a=0\cdot497100$

$E=55^\circ\ 52'\ 42''$

I

$r\sin v=0\cdot414137$

$r\cos v=0\cdot192694$

$\tan v=0\cdot221443$

$v=59^\circ\ 0'\ 45''\cdot42$

II

$\sin\frac{v}{2}\sqrt{r}=9\cdot932930$

$\cos\frac{v}{2}\sqrt{r}=0\cdot180176$

$\tan\frac{v}{2}=9\cdot752754$

$\frac{1}{2}v=29^\circ\ 30'\ 22''\cdot78$

$v=59^\circ\ 0'\ 45''\cdot57$

III

$1-e\cos E=9\cdot983913$

$\log r=0\cdot481013$

$\frac{1}{2}(v-E)=1^\circ\ 34'\ 1''\cdot68$

$v-E=3^\circ\ 8'\ 3'\ 36$

$E=55^\circ\ 52'\ 42'$

$v=59^\circ\ 0'\ 45''\cdot36.$

§ 9. The heliocentric equatorial co-ordinates obtained in equations (6) and (10) contain r and v, and therefore cannot be found unless r and v are computed first. We can, however, get rid of r and v and find the co-ordinates directly from the eccentric anomaly E. [See : Watson, p. 94 and Valentine, Vol. I., p. 415.]

Equations (10) become

$$x = k_1 r \sin A_1 \cos v + k_1 r \cos A_1 \sin v$$
$$y = k_2 r \sin B_1 \cos v + k_2 r \cos B_1 \sin v$$
$$z = k_3 r \sin C_1 \cos v + k_3 r \cos C_1 \sin v ,$$

and taking
$$r \sin v = a \sin E \cos \phi$$
$$r \cos v = a \cos E - a \sin \phi$$

and making the following substitutions

$$\left.\begin{aligned} ak_1 \sin A_1 &= l \sin L \\ ak_1 \cos \phi \cos A &= l \cos L \\ -l \sin \phi \sin L &= l_x \\ ak_1 \sin B_1 &= m \sin M \\ ak_2 \cos \phi \cos B_1 &= m \cos M \\ -m \sin \phi \sin M &= l_y \\ ak_3 \sin C_1 &= n \sin N \\ ak_3 \cos \phi \cos C_1 &= n \cos N \\ -n \sin \phi \sin N &= l_z \end{aligned}\right\} \quad \ldots \quad \ldots \quad 25\ (A)$$

we have

$$\left.\begin{aligned} x &= l \sin (E+L) + l_x \\ y &= m \sin (E+L) + l_y \\ z &= n \sin (E+N) + l_z \end{aligned}\right\} \quad \ldots \quad \ldots \quad 25\ (B)$$

where l_x, l_y, l_z are the heliocentric co-ordinates of the centre of the orbit.

These formulæ should be used only when a great number of places are to be computed; otherwise, the others previously explained should be used.

§ 10. In the parabolic orbit, we have as defined above, t = time from vertex A to P expressed in days and decimals of a day, v = the angle ASP, r = SP, Δ = EP, the distance from the Earth. The usual equations are:—

$$\tan\frac{1}{2}v + \frac{1}{3}\tan^3\frac{1}{2}v = v\,\frac{kt}{\sqrt{2}q^{\frac{3}{2}}} \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

$$r = q \sec^2\frac{v}{2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Equation (27) gives r when v is known, and equation (26) gives v from t. The given elements of the orbit include $\log q$.

Equation (26) may be put in the form

$$3 \tan \frac{v}{2} + \tan^3 \frac{v}{2} = \frac{3\,kt}{\sqrt{2}\, q^{\frac{3}{2}}}; \qquad \ldots \qquad \ldots \quad (28)$$

$\frac{3k}{\sqrt{2}}$ is a constant, and its logarithm is 8·5621876983

Also

$$75 \tan \frac{v}{2} + 25 \tan^3 \frac{v}{2} = \frac{75\,kt}{\sqrt{2}\, q^{\frac{3}{2}}} = \frac{c_o\, t}{q^{\frac{3}{2}}},$$

where $\log c_o = 9{\cdot}9601277$; and computing $\frac{c_o t}{q^{\frac{3}{2}}} = M$, the equation which is to be solved is

$$75 \tan \frac{v}{2} + 25 \tan^3 \frac{v}{2} = M, \qquad \ldots \qquad \ldots \qquad \ldots \quad (29)$$

the right hand side being known.

The solution should be obtained from Barkers' Tables, which are given in Watson's *Astronomy*.

If these Tables are not available the equation can be solved by the substitutions given in *Popular Astronomy* (1901), p. 339 and subsequently adapted by Moulton, and Picart. [See also Bauschinger, p. 13.]

$$c = \frac{2^{\frac{3}{2}}}{3k}, \quad \log c = 1{\cdot}7388423$$

$$\tan \beta = \frac{c q^{\frac{3}{2}}}{t},$$

$$\tan \gamma = \sqrt[3]{\tan \frac{\beta}{2}}$$

$$\tan \frac{v}{2} = 2 \cot 2\gamma.$$

Example.

$$t = 107{\cdot}29274$$

$$\therefore \beta = 27^\circ \quad 34' \quad 28''{\cdot}8$$

$$\therefore \gamma = 32^\circ \quad 2' \quad 57''$$

$$\frac{v}{2} = 44^\circ \quad 9' \quad 48''{\cdot}6$$

$$v = 88^\circ \quad 19' \quad 37''{\cdot}2.$$

§ 11. If great accuracy is not required, tables may be constructed according to our wants. The little table constructed by Crommelin in 1895 is useful in such cases.

Example.—To find v, where $\tan \frac{v}{2} = V$, and $V + \frac{1}{3} V^3 = 2{\cdot}434$.

$V + \frac{1}{3} V^3$	V
2 2·0	1·361
2 40	1·428
2·60	1·492

This table gives by interpolation and test-work $V = 1{\cdot}4556$.

Hence from the table of natural tangents

$$\frac{v}{2} = 55^\circ\ 31' \text{ and } v = 111^\circ\ 2'.$$

To find r with this value of v and $\log q = 9{\cdot}2831$.

$$\log q = 9{\cdot}2831 +$$

$$\text{L sec } \frac{v}{2} = 10{\cdot}2770 +$$

$$\log r = 9{\cdot}7771 +$$

§ 12. The orbits of comets having hyperbolic motion may be treated similarly by using hyperbolic functions in place of circular functions. Thus for equations (21), we have

$$\sqrt{r} \sin \frac{v}{2} = \sinh \frac{E}{2} \sqrt{a(e+1)} \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\sqrt{r} \cos \frac{1}{2} v = \cosh \frac{E}{2} \sqrt{a(e-1)}$$

$$\tan \frac{v}{2} = \sqrt{\frac{e+1}{e-1}} \tanh \frac{E}{2}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Kepler's equation in this case becomes

$$M = \text{const. } (e \sinh E - E).$$

§ 13. Now x, y, z being the heliocentric rectangular co-ordinates of the body with the equatorial system, let X, Y, Z be the co-ordinates of the Sun and ξ, η, ζ the co-ordinates of the moving body with the Earth as origin and axes parallel to the above-mentioned system of axes. X, Y, Z are tabulated in the Nautical Almanac for intervals of twelve hours. Interpolate for intermediate times.

$$\therefore \quad \xi = x + X, \ \eta = y + Y, \ \zeta = z + Z, \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

and Δ being the distance of the body from the earth,

$$\left.\begin{aligned} \xi &= \Delta \cos (\text{R. A.}) \cos (\text{dec}) \\ \eta &= \Delta \sin (\text{R.A.}) \cos (\text{dec}) \\ \zeta &= \Delta \sin (\text{dec}) \\ \Delta &= \sqrt{\xi^2 + \eta^2 + \zeta^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (31)$$

$$\therefore \tan (\text{R.A.}) = \frac{\eta}{\xi}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

$$\tan (\text{dec}) = \frac{\zeta}{\eta} \cos (\text{R.A.}) = \frac{\zeta}{\eta} \sin (\text{R.A.}) \quad \ldots \quad \ldots \quad (33)$$

also $$\sin (\text{dec}) = \frac{\zeta}{\Delta}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

These give the R. A. and dec. of the body at the time for which the computation is made. We can find when Δ is least and also the distance of the body from Jupiter.

For, if ξ', η', ζ', Δ' be similar quantities for Jupiter, these can be computed from the above equations, the R. A., dec. and distance of Jupiter being given in the Nautical Almanac.

$$\text{Hence the required distance} = \sqrt{(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2} \quad (35)$$

Similarly taking equations (10 A) in the case of the co-ordinates with reference to the ecliptic being given and R, L, B as the geocentric distance, longitude, and latitude of the Sun, as given in the Nautical Almanac and, if necessary, by interpolation; and Δ, λ, β as the geocentric distance, longitude and latitude of the moving body P, we have

$$\left.\begin{aligned} \Delta \cos \lambda \cos \beta &= \text{R} \cos \text{L} \cos \text{B} + k_1 r \sin (\text{A}_1 + v_1) \\ \Delta \sin \lambda \cos \beta &= \text{R} \sin \text{L} \cos \text{B} + k_2 r \sin (\text{B}_1' + v) \\ \Delta \sin \beta &= \text{R} \sin \text{B} + k_3 r \sin (\omega + v). \end{aligned}\right\} \quad \ldots \quad (31\ \text{A})$$

§ 14. Having completed the solution, we may sum up the different steps in the computation. The first step in an elliptic orbit is computing M from M_o and t. The second step is to compute E from M by Kepler's equation. The third step is to find r and v.

In a parabolic orbit, we obtain v from t from the cubic equation and then compute r.

Having obtained v and r, the next step in both orbits is to find the equator constants and then to find the co-ordinates of the planet or comet referred to the heliocentric equatorial system, and then the co-ordinates of the same referred to the geocentric equatorial system, and finally, the R. A. and dec. from these and also the distance.

§ 15. We have assumed, throughout that the positions of the equator and the ecliptic are fixed. The elements of the orbit are given with reference to the mean equinox of some year; that is, the position which it has at the commencement of the year. The co-ordinates X, Y, Z of the Sun in equation (30) are therefore to be taken with reference to the equinox, referred to which the elemeuts of the orbit are given, and the R. A. and dec. obtained in equations (32-34) are thus the mean R. A. and dec. of the body. In the Ephemeris generally the apparent R. A. and dec. are given. The difference between the apparent place and the mean place is usually less than one minute; and so, when great accuracy is not required, we may take the mean for the apparent place.

If we consider only the precession, leaving out the other corrections, we take the formulæ,

$$\text{correction in R. A.} = f + \frac{1}{15} g \sin (G + \alpha) \tan \delta$$

$$\text{correction in dec.} = g \cos (G + \alpha). \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

f, g, G are independent of the co-ordinates. Their values for each day of the year are given in the Nautical Almanac.

16. *Examples.*

(1) 1902 Feb. 1·5

x −1·4705207

X +0·6615245

ξ +0·8089955

Comparing (1) and (2), we have

$$D_1D_2 : E_1E_2 : F_1F_2 = \frac{p}{a} : \frac{q}{b} : \frac{r}{c},$$

which expresses the intercepts on the sides of the triangle ABC by the contact circle of S in terms of the lengths of the perpendiculars from the vertices of the medial triangle upon SS′.

y +1·3944140
Y −0·6702006
η +0·7242134
z +0·9626222
Z −0·2907456
ζ +0·6718766

△ sin (R. A.) cos (dec.) 9·859867
△ cos (R. A.) cos (dec.) 9·907946
tan (R. A.) 9·951921
(mean) R. A. 138° 9′ 54″
correction in R. A. 2 3·44
(true) R. A. 138° 11′ 57″·44, log △ = 0·106137
△ sin (dec.) 9·827290
△ cos (dec.) 0·035749
tan (dec.) 9·791540
(mean) dec +31′ 44″ 57″
correction in dec. −39″ 67
(true) dec. 31° 44′ 17″ ·63

(2) T = 1914 Oct 30·07304 G. M. T.

$\omega = 97° \ 4' \ 22''\cdot1$
$\Omega = 58 \ 43 \ 28$ } 1914·0
$i = 69 \ 4 \ 12\cdot9$

log q = 0·0524510

$\therefore$ $x = r$ [9·7797956] sin (217° 31′ 56″·4 + v)
$y = r$ [9·9082600] sin (201° 29′ 22″·1 + v) } 1914·0
$z = r$ [9·9960211] sin (117° 8′ 53″·4 + v)

Ephemeris :

	R, A.			dec.			log Δ
	H.	M.	s.	°	′	″	
March 1·5	2	42	20·1	+3	44	34	0 5904
5·5	2	43	47·0	+4	28	24	0·5919

(3) $T = 1918$ Aug. 7·906 Gr.
$\omega = 230°\ 42'\cdot18$
$\Omega = 129°\ 16'\cdot90$ | 1918·0
$i = 5°\ 27'\cdot21$

$\log q = \bar{C}\cdot19222$

1918	α 1918·0			δ 1918·0		log r	log Δ	Gr.
Dec.	H.	M.	S,	°	′			
1	4	6	38	+11	51·1	0·3403	0·0843	14 m.
5		4	17	12	1·2	0·3473	0·0994	
9		2	14	12	12·6	0 3543	0 1153	
13		0	32	12	25·4	0·3612	0·1319	
17	3	59	12	12	39·3	0 3681	0·1491	
21		38	15	12	54·3	0 3749	0·1666	
25		57	41	13	10·3	0·3816	0 1844	
29		57	30	13	27·1	0·3883	0·2024	15 m.

Brightness or visual brilliancy depends upon many causes and therefore the following formula may be taken as a rough approximation. If r, Δ be distances of the body from the sun and earth respectively at any date, and r_0, Δ_0 the same at discovery,

$$\frac{\text{Brightness at any date.}}{\text{Brightness at discovery}} = \frac{r_0^2\ \Delta_0^2}{r^2\ \Delta^2}.$$

An Extension of Feuerbach's Theorem.

By M. Bhimasena Rao, *Research Scholar, Bangalore.*

1. *Introductory.*

Feuerbach's Theorem asserts that the nine-point circle of a triangle touches its in and ex-circles. These circles may be looked upon as belonging to two classes, *viz.* (1) pedal circles, and (2) circles passing through points D, E, F on the sides of a triangle ABC, such that the triangles ABC and DEF are in perspective. It is more convenient to associate the triangle DEF with the centre S of the inconic DEF than with the centre of perspective of ABC and DEF, and call the circle DEF as the *contact* circle of S, meaning thereby the circle passing through the points of contact of the inconic of ABC, whose centre is S; for, the contact circle of isogonal conjugate points are identical—a property analogous to that of the pedal circle. Certain theorems relating to pedal circles are true for contact circles with slight changes in some cases. Thus.

(i) The contact circles of isogonal pairs through the symmedian point touch the nine-point circle. (Analogue of M'cay's Extension of Feuerbach's Theorem)

(ii) The contact circles of points on a line have a common orthogonal circle. (Analogue of Lemoyne's Theorem.)†

Feuerbach's theorem would naturally suggest the problem of drawing lines through an arbitrary point such that the pedal circles of isogonal pairs on them touch the pedal circle of the point, examining specially why there are an infinite number of solutions when the point is the circumcentre.‡

This problem has been partly considered by me already.*

In this paper it is proposed to discuss the analogous proble relating to contact circles.

*Question 201, Journal of the Indian Mathematical Society, also page 105, Vol. IV, Ibid.

† Cf. my note on "Contact Circle touching the nine-point circle" page 147, Vol. III, J. I. M. S., and also Questions 185, 202, 208, Ibid.

‡Or at infinity on an altitude of the triangle—a trivial case.

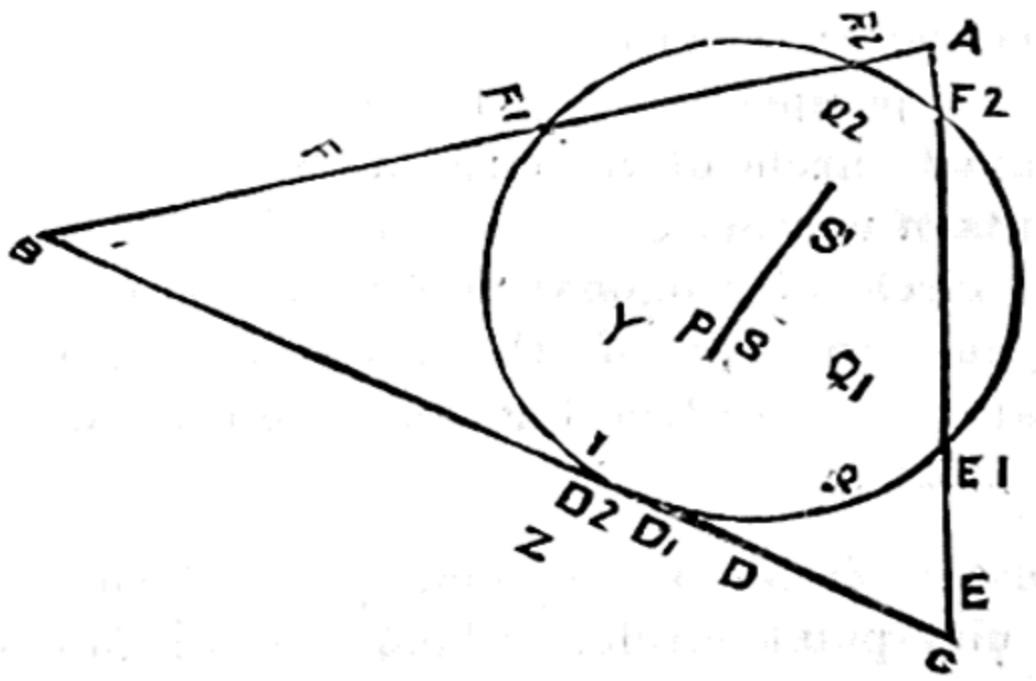
2. *Notation.*

ABC is any triangle and A′B′C′ is its medial triangle. The contact circle of isogonal conjugate points S, S′, cuts the sides of ABC in D_1D_2, E_1E_2, F_1F_2.

P is any point on the line SS′ and DEF, the contact triangle of P.

Q, Q_1, Q_2, are the centres of perspective of the triangles ABC, and DEF; ABC, $D_1E_1F_1$; ABC, $D_2E_2F_2$ respectively.

X, Y, Z, are the points of intersection of E_1F_1, E_2F_2; F_1D_1, F_2D_2; D_1E_1, D_2E_2.



3. Let p, q, r, denote the perpendiculars from A′, B′, C′ on the line SS′, and let the areal co-ordinates of S referred to the triangle ABC be f, g, h, then those of S′ are

$$\frac{a^2}{f}, \frac{b^2}{g}, \frac{c^2}{h}.$$

The equation of the inconic $D_1E_1F_1$ is

$$\sum\left\{(-f+g+h)\,x\right\}^{\frac{1}{2}}=0.$$

Hence $$BD_1=\frac{a(b-g+h)}{2f}.$$

Similarly $$BD_2=a\left\{\frac{a^2}{f}-\frac{b^2}{g}+\frac{c^2}{h}\right\}\Big/\frac{2\omega}{f}$$

$$\therefore D_1D_2=\left\{g(c^2f^2-a^2h^2)+h(a^2g^2-b^2f^2)\right\}/(2\,afgh). \quad \ldots \quad (1)$$

The equation of the line SS′ is

$$\Sigma\, f(c^2g^2-b^2h^2)=0.$$

The ratios of the perpendiculars from A′, B′, C′ on SS′ are as

$$[g\,(a^2h^2-c^2f^2)+h(b^2f^2-a^2g^2)], \text{ etc.} \quad (2)$$

It will be later on useful to express these same intercepts also in terms of the normal co-ordinates of any point P on SS′ and the lengths of the tangents from D, E, F to the contact circle of S. This is easily done as follows.

Since P is a point on SS′, it follows that Q, Q_1, Q_2 lie on a circum-conic of ABC.

$$\therefore \quad B(AQQ_1Q_2) = C(AQQ_1Q_2).$$

$$\therefore \quad (AEE_1E_2) = (AFF_1F_2).$$

$$\therefore \quad \frac{AE.E_1E_2}{AE_1.EE_2} = \frac{AF.F_1F_2}{AF_1.FF_2},$$

and

$$\frac{AE.E_1E_2}{EE_1.AE_2} = \frac{AF.F_1F_2}{FF_1.AF_2}.$$

Multiplying

$$\frac{(AE.E_1E_2)^2}{AE_1.AE_2.EE_1.EE_2} = \frac{(AF.F_1F_2)^2}{AF_1.AF_2.FF_1.FF_2}.$$

But $AE_1.AE_2 = AF_1.AF_2$ being equal to the square of the tangent from A to the contact circle $E_1E_2F_1F_2$ and $EE_1.EE_2 = t_2^2$ and $FF_1.FF_2 = t_3$ where t_1, t_2 and t_3 denote the lengths of the tangents from D, E, F to the contact circle of S.

$$\therefore \quad \frac{AE.E_1E_2}{t_2} = \frac{AF.F_1F_2}{t_3}.$$

Since AP bisects EF, m AE. $= n$ AF. if l, m, n denote the normal co-ordinates of P with respect to ABC.

Hence finally $D_1D_2 : E_1E_2 : F_1F_2 = lt_1 : mt_2 : nt_3$, for all positions of P on the line SS′.

4. The results in the preceding paragraph can be immediately used to effect the solution of the problem proposed. Supposing that P is the given point through which it is required to draw a line such that the contact circle of isogonal conjugate points on it touch the contact circle of P, and denoting the sides of the triangle DEF by a', b', c', we should have $a't_1 \pm b't_2 \pm c't_3 = 0$. (1)

Also we have seen that

$$D_1D_2 : E_1E_2 : F_1F_2$$

$$= \frac{p}{a} : \frac{q}{b} : \frac{r}{c} = lt_1 : mt_2 : nt_3. \quad \ldots \quad \ldots \quad (2)$$

Hence substituting the values of t_1, t_2, t_3 in (1), the condition of tangency of the contact circles of P and S is seen to be

$$\frac{a'p}{al} \pm \frac{b'q}{bm} \pm \frac{c'r}{cn} = 0 \; ; \quad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

since the areal equation of the line PSS′ is $px+qy+rz=0$, result (3) shows that PSS′ should pass through one of the four points

$$\left(\frac{a'}{al}, \pm \frac{b'}{bm}, \pm \frac{c'}{cn} \right)$$

which are the vertices of a quadrangle having A′B′C′ as the diagonal triangle. These points are determined when P is given, and the problem is accordingly solved. Four lines can in general be drawn through any point P satisfying the conditions of the problem, but when P coincides with one of the four points $\left(\frac{a'}{al}, \pm \frac{b'}{bm}, \pm \frac{c'}{cn} \right)$, the problem admits of an infinite number of solutions.

Also, when P is the symmedian point, it is easy to verify that it coincides with $\left(\frac{a'}{al}, \pm \frac{b'}{bm}, \pm \frac{c'}{cn} \right)$.

5. Consider now the conic ABC QQ_1Q_2 and the tangent to it at Q.

The line EF passes through X the point of intersection of E_1F_1 and E_2F_2, because Q lies on the conic ABC Q_1Q_2. From the triangle AEF and the transversal E_1F_1, we have

$$\frac{AF_1}{F_1F} \cdot \frac{FX}{XE} \cdot \frac{EE_1}{E_1A} = 1.$$

Similarly,
$$\frac{AF_2}{F_2F} \cdot \frac{FX}{XE} \cdot \frac{EE_2}{E_2A} = 1.$$

Multiplying

$$\frac{AF_1 . AF_2}{FF_1 . F_2F} \left(\frac{FX}{XE} \right)^2 \frac{EE_1 . EE_2}{E_1A . E_2A} = 1.$$

Since $F_1F_2E_2E_1$ are concyclic, $AF_1 . AF_2 = E_1A . E_2A$; and $F_2F . F_1F = t_3^2$; $EE_1 . EE_2 = t_2^2$.

$$\therefore \quad \frac{FX}{XE} = \frac{t_3}{t_2},$$

showing that DX, EY FZ concur at the point whose areal co-ordinates with respect to DEF are $(1/t_1, 1/t_2, 1/t_3)$.

Again X is the pole of BC with respect to the conic ABC QQ_1Q_2. The tangent at Q to the conic and the line QX are harmonic conjugates with respect to QB, QC, *i.e.*, QE, QF. If this tangent cuts EF in X′, we have

$$\frac{FX}{XE} = \frac{FX'}{EX'}$$

Hence this tangent is the triangular polar of $(1/t_1, 1/t_2, 1/t_3)$ with respect to DEF. Its equation is therefore $xt_1+yt_2+zt_3=0$ in areals. Consequently the tangents t_1, t_2, t_3 from D, E, F to the contact circle of S are as the perpendiculars from D, E, F to the tangent to the conic.

Since we should have $a't_1 \pm b't_2 \pm c't_3=0$ when the circle DEF touches the circle $D_1E_1F_1$, the above result shows that the tangent at Q should in that case pass through the points $(a', \pm b', \pm c')$, which are the in and excentres of the triangle DEF.

When P is the symmedian point, Q is the orthocentre and DEF the orthocentric triangle. Since Q is now the incentre of DEF, it is clear why there are an infinite number of solutions in this case.

Since the lines through Q are thus determined, it now remains to draw the corresponding lines through P.

Let R be any point on the conic ABCQ touching the line joining Q to the in or an excentre of the triangle DEF. If AR, BR, CR cut the sides of ABC at D_3, E_3, F_3, the centre of the in-conic $D_3E_3F_3$ is a point on the required line. We shall take R to coincide with A and the in-conic D_3, E_3, F_3, degenerates into the tangent at A to the conic, and this tangent is the fourth harmonic of AX with respect to AB, AC showing that X′, the point of intersection of the line joining Q to the in or excentres of DEF with EF, is a point on the tangent at A.

The following construction will now become obvious.

Let P be the given point and D, E, F, the points of contact of the inconic of ABC whose centre is P. Join BE, CF intersecting at Q, and let the line joining Q to the in or an excentre of DEF cut EF in X′. If AX′ cuts the side B′C′ of the medial triangle of ABC in V, then the contact circle of isogonal conjugate points on PV touches the contact circle of P.

6. *The points of contact.*

If QX′ in the construction given above cuts the remaining sides of DEF in Y′, Z′, it is obvious that the lines joining Y′ and Z′ to the middle points of the arcs DF and DE respectively intersect at the point of contact. That these lines meet on the circle DEF follows from Pascal's theorem. If T be the point of intersection, it will be evident on drawing the figure that DT : ET : FT $= t_1 : t_2 : t_3$.

Let the normal co-ordinates of Q referred to the triangle DEF be l, m, n. The equation of the line joining Q to the incentre of DEF is

$$\alpha(m-n)+\beta(n-l)+\gamma(l-m)=0.$$

The normal co-ordinates of T are

$$\left(\frac{a'}{m-n}, \frac{b'}{n-l}, \frac{c'}{l-m}\right)$$

Those of T_1 corresponding to the D-excentre are

$$\left(\frac{a'}{m-n}, \frac{b'}{n+l}, -\frac{c'}{l-m}\right)$$

Hence TT_1 is

$$(m-n)^2\frac{\alpha}{a'}+(n^2-l^2)\frac{\beta}{b'}-(l^2-m^2)\frac{\gamma}{c'}=0.$$

Similarly T_2T_3 (corresponding to the remaining excentres) is

$$-(m+n)^2\frac{\alpha}{a'}-(n^2-l^2)\frac{\beta}{b'}+(l^2-m^2)\frac{\gamma}{c'}=0.$$

Hence TT_1 and T_2T_3 intersect on EF at the point L, *viz.*

$$\left(0, \frac{b'}{n^2-l^2}, \frac{c'}{l^2-m^2}\right).$$

If M, N be the remaining diagonal points of the quadrangle $TT_1T_2T_3$, then DL, EM, FN concur at the point

$$\left(\frac{a'}{m^2-n^2}, \frac{b'}{n^2-l^2}, \frac{c'}{l^2-m^2}\right)$$

on the circle DEF. This point is also on the nine-point-circle of ABC being the centre of the rectangular hyperbola ABCQ

$$\Sigma\,\alpha^2(m^2-n^2)=0.$$

It may have been noticed that the diagonal points of the quadrangle $TT_1T_2T_3$ were found to be on the sides of the triangle DEF. The theorem will thus enable us to see the reason for the same.

The T—tangents touch also the nine-point conic ABCQ.

When Q is the median point of ABC, this will lead to Mr. S. Narayanan's Question 248, *viz.*, the F—tangents touch the maximum inscribed ellipse of the triangle.

This theorem itself is easily proved. For, consider a variable line through P. The contact circles of all points on this line have a common orthogonal circle *viz.* the director circle of the corresponding conic ABCQ. If π be the centre of this conic the locus of π is evidently the nine point conic ABCQ. If S be a point on a line through P, the radical axis of the contact circles of S and P cuts the nine point conic ABCQ in two points, π and π', corresponding to PS and PS'. When P, S, S' are in a line, it is evident that the radical axis touches the nine point conic ABCQ, and hence the theorem.

If then Σ is the fourth point of intersection of the circle DEF and the nine point conic ABCQ, *i.e.* the centre of the rectangular hyp. ABCQ, it follows from a well known theorem that the three quadrangles DEFΣ, $TT_1T_2T_3$ and $\pi\pi_1\pi_2\pi_3$ where the π's are the centres of the common orthogonal circles corresponding to the P—lines, have a common diagonal triangle.

It does not appear that any special properties relating to the T-points can be given when P is an arbitrary point; on the other hand, when the contact circle of P is also a pedal circle, many theorems involving the nine point circle and the in and excircles are capable of extension. It is not the purpose of this paper to deal with pedal-contact circles, but we shall give below an investigation of the conditions under which the isogonal conjugate points on the P—lines are such that, taken two by two (in a certain order) they are collinear with the vertices of the triangle, in other words, form a pair of *Steinerian quadruples* as Mr. Emch calls them in his *Projective Geometry*.

Let S, S'; S_1 S'_1; $S_2S'_2$; $S_3S'_3$ be the isogonal conjugate points on the P—lines. The collinearity of the triads of points ASS_1, BSS_2, CSS_3 will, we shall assume, be secured when the rectangular hyperbola through P self-conjugate with respect to ABC meets the P—lines on points forming a quadrangle having ABC as the diagonal triangle. This will be the case if the P—lines taken two by two form conjugate lines of the three systems of involutions determined by the lines PA, PB, PC, PP' where P' is the isogonal conjugate of P. Let the centre of the common orthogonal circle of the contact circle of points on a line

be termed for brevity as the *contact-pole* of the line. It may easily be shown that the anharmonic ratio of any four lines through P is equal to that of their contact-poles. The contact-poles of the P-lines are π, π_1, π_2, π_3 and those of PA, PB, PC are the middle points of the sides of ABC. Let Ω be the contact-pole of PP'. Then the statement that the π-points taken two by two form conjugate pairs belonging to the three systems of involution determined by A'B'C'Ω amounts to saying that the quadrangles formed by the π-points and A'B'C'Ω have a common diagonal triangle. Hence the necessary condition is that the diagonal points of the quadrangle formed by π or T points should be on the sides of the medial triangle A'B'C'. This condition can be shown to be sufficient.

Exactly the same points A'B'C'Ω would have been determined if we had started with P' instead of P.

It follows as a consequence of the obove condition, that the common diagonal triangle of π, T-points, DEFΣ, and A'B'C'Ω is also the common diagonal triangle of π', T'-points and D'E'F'Σ' where the dashes indicate the corresponding points for P'. If LMN be the diagonal triangle of the seven quadrangles, since EF, E'F' and B'C, intersect at L and similar triads of lines at M and N, the centres of perspective of ABC and DEF, D'E'F' A'B'C', viz, Q, Q', G (the median point) lie on a circumconic of ABC and therefore P, P', G are collinear. Since the join of isogonal pairs P, P', passes through the median point, it follows that the inconic of ABC whose centre is P touches the sides at points where the normals are concurrent, or in other words, the circle DEF is a pedal circle.*

7. *Conclusion.*

A few words on one or two points may not be out of place. The analogy between pedal and contact circles seems to be remarkable. The result of § 6 is true for pedal circles, *i.e.*, if the pedal circle of P is also a contact circle, and the four lines are drawn through P such that the pedal circle of isogonal conjugate points on them touch the pedal circle of P, the four pairs of these isogonal conjugate points form a pair of steinerian quadruples. It is this property which has enabled me to extend Mr. V. Ramaswami Aiyar's result "The director circles of the inconics through the circumcentre and the orthocentre touch the nine point circle at the Feuerbach points" as proposed by me in

* Vide: Question 430, Journal of the Indian Mathematical Society.

Question 877. Yet it does not appear that a general relation can be set up between the two kinds of circles, so that the properties of either kind could be deduced from known properties of the other, avoiding the necessity of a separate investigation. Another point which I would like to mention is that many properties of the nine point circle and the in and ex-circles are capable of extension to pedal-contact circles. I am not aware whether the properties of pedal-contact circles have been exhaustively studied.

Postscript.

Mr. V. Ramaswami Aiyar, M. A., has stated in Question 863 of the Journal of the Indian Mathematical Society the following remarkable theorem:—

The pedal circle of any point P with respect to ABC cuts the nine-points circle at an angle equal to the complement of the sum of the angles PAB, PBC, PCA.

This property is not true for the contact circle, but it may be put in the following form which holds good for pedal and contact circles:—

The pedal circle of any point with respect to a triangle cuts the nine points circle at the sum of the angles at which it intersects the sides of the triangle.

The following property suggested by Feuerbach's theorem may be found interesting. My proof of it is not yet complete.

"Of the system of pedal-contact circles, each circle touches four circles which latter are also touched by three more circles of the system."

* Vide, page 132, Vol. IX. Journal of the Indian Mathematical Society. The extension referred to is.

"If P be a point whose pedal triangle with respect to A B C is in perspective with A B C, show that associated with P, there is another point Q on the line joining P with its isogonal conjugate, such that the director circles of the four inconics of A B C passing P and Q all touch the pedal circle of P"

SHORT NOTES.

Note on Question 653.

Q. 653. (N. B. MITRA) :—To find a cube the sum of whose divisors is a perfect square.

One solution in positive integers may be found in the following manner:

Let x^3 be the required cube and suppose x to be prime. Then the sum of the divisors of x^3 is $(1+x+x^2+x^3)$ which mustbe a square, say u^2, so that

$$1+x+x^2+x^3=u^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let $u=nx-1$, where n is an integer.

Substituting in (1) and simplifying,

$$x^2+x(1-n^2)+1+2n=0.$$

$$\therefore \quad x=\tfrac{1}{2}\{n^2-1\pm\sqrt{(n^4-2n^3-8n-3)}\} \quad \ldots \quad (2)$$

Since x is rational the expression under the radical sign must be a perfect square. Let $n^4-2n^3-8n-3=v^2$. $\ldots \quad \ldots \quad \ldots$ (3)

Then $$(n^2-1)^2-4(2n+1)=v^2$$

or $$(n^2-1-v)(n^2-1+v)=4(2n+1) \quad \ldots \quad \ldots \quad (4)$$

Now from (2) & (3), $x=\frac{1}{2}(n^2=1\pm v)$, so that, since x is an integer, n^2-1 and v must be either both odd or both even; that is, both their sum and difference are even. Hence from (4), it is evident, we may put $n^2-1-v=2$ and $n^2-1+v=4n+2$, whence $v=2n$ and $n^2-2n-3=0$. Therefore $n=3$, $v=6$ or $n=-1$, $v=-2$. Therefore $x=7$ or 1 and when $x=7$, $u=20$.

If a and b be two different prime values of x then $x=ab$ is also a solution of the problem. For, the sum of the divisors of a^3b^3 is $(1+a+a^2+a^3)(1+b+b^2+b^3)$, which is a perfect square since the two factors of it are perfect squares. It appears, however, that 7 is the only value of x; at any rate, there is no other prime value of it less than two billions, as the following investigation will show.

Since $1+x+x^2+x^3$ or $(1+x)(1+x^2)$ is to be a square, we may put

$$1+x=ka^2 \ldots\ldots (5)$$

$$1+x^2=kb^2 \ldots\ldots (6)$$

where k, , ba are integers.

$$x \equiv -1 \pmod{k};$$

$$\therefore \quad x^2 \equiv 1 \pmod{k};$$

$$x^2+1 \equiv 2 \pmod{k}.$$

But from (6) $x^2+1 \equiv 0 \pmod{k}$.

Hence $2 \equiv 0 \pmod{k}$.

Therefore k is either 1 or 2. If k is one, then from (6), $x^2+1=b^2$, whose only solution in positive integers is $x=0$, $b=1$.

If $k=2$, we have $1+x=2a^2$ & $1+x^2=2b^2$. Eliminating x we have, $a^4+(a^2-1)^2=b^2$, Hence b is the hypotenuse of a right angled triangle whose perpendicular sides differ by unity; besides, the greater of the two perpendicular sides is a perfect square. Now, forming successively the sides of a right-angled triangle whose perpendicular sides differ by 1 and whose sides are integers, we have the following table:—[The sides are obtained from the formulæ:—

$$(1)\ \tfrac{1}{4}\{(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}+2\},$$

$$(2)\ \tfrac{1}{4}\{(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}-2\},$$

$$\text{or } (3)\ \frac{1}{2\sqrt{2}}\{(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n-1}\}\].$$

Greater side.		Smaller side.		Hypotenuse.
1	...	0	...	1
4	...	3	...	5
21	...	20	...	29
120	...	119	...	169
697	...	696	...	985
4060	...	4059	...	5741
23661	...	23660	...	33461
137904	...	137903	...	195025
803761	...	803760	...	1136689
4684660	...	4684659	...	6625109
27304197	...	27304196	...	38613965
159140520	...	159140519	...	225058681
927538921	...	927538920	...	1311738121
5406093004	...	5406093003	...	7645370045
31509019101	...	31509019100	...	44560482149
183648021600	...	183648021599	...	259717522849

Now of the 16 numbers in the first column, the only ones which are squares are 1 and 4 and these give $a=1$ or 2 and $b=1$ or 5; so that $x=1$ or 7. Hence 7^n is a solution of the problem. Also, since, $x=2a^2-1$, the above table shows that there is no other value of x less than $2\times183648021600-1$ or 367296043199. In fact, the next number in the first column is greater than 10^{12}. Consequently the next value of x must be at least greater than 2 billions.

N. B. Mitra.

Note on the continued fraction in Q. 713.

Q. 713. (Communicated by K. V. Anantanarayana Sastri):—Find the value of the infinite continued fraction

$$\cfrac{1}{1+}\ \cfrac{1^4}{3+}\ \cfrac{2^4}{5+}\ \cfrac{3^4}{7+}\ \ldots+\cfrac{n^4}{2n+1+}\ \ldots$$

Among the solutions that the editor has received, the following method by reduction given by Messrs. R. J. Pocock and K. A. Erady, (both since deceased), is purely algebraical. It appears to me however that the chief interest of the present example lies in the derivation of the continued fraction as an "equivalent" of some asymptotic series. In addition to giving a posthumous publication of the solutions just mentioned, we shall in this note proceed from the given fraction to the series, from where we will naturally be led to a functional equation, if we desire to sum the series.

Since, with the customary notation,

$$p_{n+1}=(2n+1)\,p_n+n^4\,p_{n-1},$$

we have $p_{n+1}-p_n.\,(n+1)^2=-n^2\,[p_n-p_{n-1}.\,n^2]$.

$$\therefore\quad p_{n+1}-p_n.\,(n+1)^2=(-)^{n-1}(n\,!)^2(p_2-p_1.\,2^2)=(-)^n(n\,!)^2.$$

Writing this in the form

$$\frac{p_{n+1}}{[(n+1)\,!]^2}-\frac{p_n}{(n\,!)^2}=\frac{(-)^n}{(n+1)^2},$$

we obtain at once

$$\frac{p_{n+1}}{[(n+1)!]^2}=1-\frac{1}{2^2}+\frac{1}{3^2}\ \ldots\ \text{to}\ (n+1)\ \text{terms.}$$

Similarly,

$$q_{n+1}-q_n(n+1)^2=(-)^{n-1}(n\,!)^2(q_2-q_1.\,2^2)=0$$

and therefore

$$q_{n+1}=[(n+1)\,!]^2.$$

The continued fraction therefore equals

$$\lim_{n=\infty}\ \frac{p_{n+1}}{q_{n+1}}=1-\frac{1}{2^2}+\frac{1}{3^2}\ldots\ldots=\frac{\pi^2}{12}.$$

Let us however consider more generally,

$$f(x) \equiv \frac{x}{1+}\ \frac{1^4x^2}{3+}\ \frac{2^4x^2}{5+}\ldots+\frac{n^4x^2}{(2n+1)+}\ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and write

$$f(x) \equiv \frac{x}{1-}\ \frac{a_1a_2x^2}{1-}\ \frac{a_3a_4x^2}{1-} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

to have to deal with only a continued fraction of the second class.

We need only take

$$\left.\begin{aligned} a_{2n} &= -\frac{n^2}{2n+1} \\ a_{2n+1} &= \frac{(n+1)^2}{2n+1}, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

Now we can easily verify that the $2n^{th}$ convergent of

$$\frac{1}{1-}\ \frac{a_1x}{1-}\ \frac{a_2x}{1-} \quad \ldots \quad \ldots \quad (4)$$

is identical with the n^{th} convergent of

$$\frac{1}{[1-a_1x]}-\frac{a_1a_2x^2}{[1-(a_2+a_3)x]}-\frac{a_3a_4x^2}{[1-(a_4+a_5)x]}\ldots \quad \ldots \quad \ldots \quad (5)$$

For, from (4)

$$P_{2n} = P_{2n+1} - a_{2n-1}\,x\,P_{2n-1}$$

and two other equations. Eliminating between these P_{2n-1} and P_{2n-3} we have

$$P_{2n} = [1-(a_{2n-2}+a_{2n-1})x]\,P_{2n-2} - a_{2n-3}a_{2n-2}x^2\,P_{2n-4}$$

which is the recurring formula for (5).

Thus (4) and (5) are equivalent, and we will donote by (A) the transformation implied by them.

Our purpose being to obtain a functional equation for $f(x)$ in its simplest form, it is obviously simplest to take

$$1 = a_1 = a_2+a_3 = a_4+a_5 = \ldots \quad \ldots \quad \ldots \quad (5')$$

which is also consistent with (3).

Moreover in order to have

$$E \equiv \frac{1}{1-}\ \frac{a_1x}{1-}\ \frac{a_2x}{1-}\cdots\cdots \qquad \ldots \qquad \ldots \qquad \ldots \quad (6)$$

equivalent to

$$F \equiv 1 + \frac{b_1x}{1-}\ \frac{b_2x}{1-}\ \frac{b_3x}{1-}\cdots \qquad \ldots \qquad \ldots \qquad \ldots \quad (7)$$

We should have, generally

$$\left.\begin{array}{l} b_1 = a_1 \ : \quad b_2 = a_1 + a_2 \ : \\ b_2b_3 = a_3a \ : \quad b_3 + b_4 = a_3 + a_4 \ : \end{array}\right\} \text{etc.} \qquad \ldots \qquad \ldots \qquad \ldots \quad (8)$$

Let us call (B) the transformations implies by (6), (7), (8.)

This is easily seen for,

$$E = \frac{1}{1 - xE_1} = 1 + \frac{a_1x}{\frac{a_1}{E_1} - a_1x}$$

where $E_1 = \frac{a_1}{1-}\ \frac{a_2x}{1-}\ \frac{a_3x}{1\ldots}$

$$= \frac{a_1}{1 - a_2x -}\ \frac{a_1a_3x^2}{1 - (a_3 + a_4)x^2}\ - \ldots$$

on applying the transformation (A).

Applying the same transformation to (7) also, it is seen equivalent to

$$1 + \frac{b_1x}{1 - b_2x -}\ \frac{b_2b_3x^2}{1 - (b_3 + b_4)x -} \quad \ldots$$

Hence the conditions (8) and the transformation (B).

Substituting the values (3) in (8), we find

$$b_{2n} = -b_{2n+1} = \frac{n\,(n+1)}{2n+1} \quad \text{with } a_1 = b_1 = 1.$$

With these values we can easily verify that

$$\frac{b_1}{1-}\ \frac{b_2\,x}{1-}\ \frac{b_3\,x}{1-}\cdots$$

and $2-\dfrac{b_1}{1+}\ \dfrac{b_2\,x}{1+}\ \dfrac{b_3\,x}{1+}\cdots$ are equivalent.

for $b_1=1$ and $b_{2n}+b_{2n+1}=0$.

This we will call transformation (C).

Now from the main question

$$f(x)\equiv\frac{x}{1-}\ \frac{a_1\,a_2\,x^2}{-\ 1}\ \frac{a_3\,a_4\,x^2}{1-}\ \cdots$$

$$\therefore \frac{1}{x}f\left(\frac{x}{1-x}\right)=\frac{1}{(1-x)}-\frac{a_1\,a_2\,x^2}{(1-x)}-\frac{a_3\,a_4\,x^2}{(1-x)}-\cdots$$

$$=\frac{1}{1-}\ \frac{a_1\,x}{1-}\ \frac{a_2\,x}{1-}\ \cdots \qquad \text{from (5′) and (A)}$$

$$=1+\frac{b_1\,x}{1-}\ \frac{b_2\,x}{1-}\ \frac{b_3\,x}{1-}\cdots \qquad \text{by (B)}$$

$$=1+2\,x-\frac{b_1\,x}{1+}\ \frac{b_2\,x}{1+}\ \frac{b_3\,x}{1+}\cdots \qquad \text{by (C)}$$

$$=2\,x+1-\frac{b_1\,x}{1+}\ \frac{b_2\,x}{1+}\ \frac{b_3\,x}{1+}\cdots$$

$$=2\,x+\frac{1}{1+}\ \frac{a_1\,x}{1+}\ \frac{a_2\,x}{1+}\ \frac{a_3\,x}{1+}\cdots \qquad \text{by (B)}$$

$$=2\,x+\frac{1}{1+a_1\,x}-\frac{a_1\,a_2\,x^2}{1+(a_2+a_3)x}-\frac{a_3\,a_4\,x^2}{1+(a_4+a_5)x}\cdots \qquad \text{by (A)}$$

$$=2\,x+\frac{1}{(1+x)}-\frac{a_1\,a_2\,x^2}{(1+x)}-\frac{a_3\,a_4\,x^2}{(1+x)}-\cdots \qquad \text{from (5′).}$$

$$=2x+\frac{1}{x}\,f\left(\frac{x}{1+x}\right).$$

Thus we have the functional equation

$$f\left(\frac{x}{1-x}\right)-f\left(\frac{x}{1+x}\right)=2x^2.$$

To solve this put

$$f(x)=\int_0^\infty e^{-t/x}\,\phi(t)\,dt$$

Then,

$$f\left(\frac{x}{1-x}\right)-f\left(\frac{x}{1+x}\right)=2\int_0^\infty e^{-t/x}\sinh t\phi(t)dt=2x^2,$$

But

$$2x^2=2\int_0^\infty t\,e^{-t/x}dt.$$

And therefore by a well-known formula of Lerch

$$\phi(t)=t\ \text{cosech}\ t.$$

Hence finally

$$\frac{x}{1+}\frac{1^4x^2}{3+}\cdots\frac{n^4x^2}{(2n+1)+}\cdots\cdots=\int_0^\infty t\ \text{cosech}\ t\ e^{-t/x}\,dt.$$

Putting $x=1$, we identify it easily with the value $\frac{\pi^2}{12}$ obtained before.

The reverse process of expanding the integral (which is of the 'Borel' form) into an Asymptotic series and expressing that series as a continued fraction is also instructive. We content ourself by simply giving references to

Rogers : *Proc. Lond. Math. Soc.* Series 2 Vol. 4.

or

Oskar Perron. *Die Lehre von den Kettenbruchen* (Teubner : 1913) pp. 380—392.

or

My solution to Q. 541, J. I. M. S. Vol. VIII, pp. 17—20.

K. B. Madhava.

Astronomical Notes.

1. *Comets.* The fourth comet for the year (1919 d) was discovered by Mr. Sasaki at Kyoto, Japan, on October 25. Its position at that time was R-A 20 h. 17 m. and Decl. 27° 11′ S. It was reobserved by M. Schaumasse at Nice and found identical with Finlay's periodic comet—the period being $6\frac{1}{2}$ years.

Comet 1911 VII which was due to return to perihelion early in November was discovered by M. Schaumasse; the magnitude at the time of discovery was about 12-0 and the comet is now getting fainter.

New elements have been computed for the comet 1919 b. (Brorsen-Metcalf) from the available observations which indicate a shortening of the period from 72 years to 42. The comet appears to have completed two revolutions since 1847, so that it should belong to the family of Uranus and not to that of Neptune, in which it has hitherto been classified.

2. All the planets except Uranus are on one side of the Sun. Their positions on the 1st January 1920 will be as follows:—

Planet.	R.A. H.	R.A. M.	Decl.	Constellation.
Mercury ...	17	18	22°·4 S	Scorpio
Venus ...	15	40	16° 9 S	Scorpio
Mars ...	13	5	4°·8 S	Virgo
Jupiter ...	9	19	16° 5 N	Cancer
Saturn ...	10	55	8° 9 N	Leo
Uranus ...	22	5	12°·5 S	Capricorn
Neptune ...	8	54	17°·5 N	Cancer.

The Sun's position on the same day will be

R.A. 18 h. 42 m. and decl. 23°5′ S.

3. An interesting minor planet was discovered in March and provisionally designated 1919 F.D. This has the same mean motion as Jupiter and is thus the sixth member of the Trojan group. Its mean longitude is 60° greater than that of Jupiter.

4. At a joint meeting of the Royal Society and the Royal Astronomical Society, the results obtained at the total Solar eclipse of May 29, were reported and discussed. Two sets of photographs were obtained at Sobral, N. Brazil, one with a 13-inch objective and the other with a 4-inch, while a third set was secured at the station Principe Island. The results appear to show conclusively that there exists a gravitational effect, the amount of the shift at the Sun's limb obtained in each case being as follows:—

	Amount of Shift.	Probable Error.	Instrument.
I	0″·93 or 0″·99	± 0″·3	13″ at Sobral.
II	1″·98	± 0″·12	4″ ,,
III	1″·60	± 0″·3	12″ at Principe.

The mean of the last two results agrees very closely with Einstein's predicted value 1″·75. His law of gravitation has already explained the motion of the perihelion of Mercury and by the preceding test at the total Solar eclipse, the *law* may be considered as confirmed.

NIZAMIAH OBSERVATORY,
HYDERABAD, *December* 1919.

T. P. Bhaskara Sastri.

SOLUTIONS.

Question 695.

(R. Srinivasan M. A.) :—From a point P on a sphere four straight lines PA, PB, PC, PD are drawn to meet it in A,B,C,D such that $A\hat{P}B = B\hat{P}C = C\hat{P}D = D\hat{P}A$., show that $PA + PC = PB + PD$.

Solution by N. Sankara Aiyar.

The equation of the sphere may be taken

$$x^2 + y^2 + z^2 + 2fy + 2gx + 2hz = 0$$

the origin being at P.

From the given condition PA, PB, PC, PD are seen to be edges of a square pyramid with vertex at P. The planes APC and BPD are therefore perpendicular. They may be taken as $y = 0$, $x = 0$.

Then

$$AP \text{ is } y = 0, \quad x = mz,$$

$$CP \text{ is } y = 0, \quad x = -mz,$$

$$BP \text{ is } x = 0, \quad y = mz,$$

$$DP \text{ is } x = 0, \quad y = -mz,$$

where $m = \tan \frac{APC}{2}$.

The co-ordinates of A are given by

$$y = o, \; z^2(1 + m^2) + 2mgz + 2hz = 0.$$

i.e.

$$z = -2\frac{(h + mg)}{1 + m^2}, \; x = -2m\frac{h + mg}{1 + m^2}.$$

$$\therefore \quad AP^2 = 4\frac{(k + mg)^2}{1 + m^2}.$$

Similarly $CP^2 = 4\frac{(h - mg)^2}{1 + m^2}$.

$$\therefore \quad AP + CP = \frac{4h}{1 + m^2}.$$

The co-ordinates of B are similarly given by

$$x=0,\ z^2(1+m^2)+2mfz+2hz=0.$$

i.e. $$z=-2\frac{h+mf}{1+m^2},\ y=-2m\frac{h+mf}{1+m^2};$$

and $$BP+DP=\frac{4h}{1+m}$$

Hence $$AP+CP=BP+DP.$$

Question 870.

(S. RANGANATHAN) :—In a rectangular hyperbola A_1 is any point on the line bisecting at right angles the semi-transverse axis; A_2 is a second point on the same determined by drawing A_1P parallel to one asymptote to cut the curve in P, and PA_2 parallel to the other asymptote. If A_3 A_4 are similarly formed from A_2, A_3 respectively, show that A_4 coincides with A_1.

Solution by Hemraj, S. V. Venkatachala Iyer and S. Muthukrishnan.

Let P, Q, R he the points in which A_1P, A_2Q, A_3R cut the rectangular hyperbola $xy=c^2$, the lines being parallel to y—axis.

If P be $(ct_1, c/t_1)$ and Q $(ct_2, c/t_2)$, then A_2 is $(ct_2, c/t_1)$.

Since A_2 is on the right bisector of the semi-transverse axis; which is $$y-c/2=-x+c/2,$$

it follows $$t_2=1-\frac{1}{t_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Evidently A_3 is $\{c(1-1/t_2), c/t_2\}$ and consequently R is $\{c(1-1/t_2), ct_2/(t_2-1)\}$.

The point of intersection of PA_1 and RA_4 $[ct_1, ct_2/(t_2-1)]$ lies on the right bisector, if

$$\frac{ct_2}{t_2-1}-\frac{c}{2}=-ct_1+\frac{c}{2}.$$ *i.e.* if $t_2=1-\frac{1}{t_1}$, which is (1).

Hence the result.

QUESTIONS FOR SOLUTION.

1080. (K. J. SANJANA, M.A.):—An elliptic cylinder of eccentricity e rests on a horizontal plane and against a plane inclined at an angle 2α to the horizon, touching them along lines parallel to and equidistant from their line of intersection. If the coefficient of friction (μ) be the same for each plane and the cylinder be just on the point of slipping, prove that μ is the positive root of the equation

$$\mu^2 (\tan \alpha - e^2 \sin \alpha \cos \alpha) + \mu = e^2 \sin \alpha \cos \alpha.$$

1081. (K. J. SANJANA, M.A.) :—If a, b, c, be real unequal quantities whose sum is zero, prove that the equation

$$4 (a^2+b^2+c^2) x^3 + 18 abcx^2 - abc (a^2+b^2+c^2) = 0$$

has two imaginary or complex roots.

1082. (N. P. PANDYA) :—Two concentric ellipses intersect at A, B, C, D and the director circle of one ellipse coincides with the auxiliary circle of the other. Find the locus of the centre of gravity of the quadrilateral ABCD.

1083. (N. P. PANDYA) :—A circle with centre O is fixed in a plane α. Find the condition that another circle in a plane β may be projected on α into an ellipse with O as one of its vertices and with its centre C on the circumference of the circle O.

1084. (HEMRAJ) :—Sum up the following series

$$\left(\frac{1}{1^3}+\frac{1}{2^3}\right)+\frac{1}{2}\left(\frac{1}{3^3}+\frac{1}{4^3}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{1}{5^3}+\frac{1}{6^3}\right)$$

$$+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{7^3}+\frac{1}{8^3}\right)+\frac{1\,3\cdot5\cdot7}{2\cdot4\,6\cdot8}\left(\frac{1}{9^3}+\frac{1}{10^3}\right)+\text{to infinity}$$

1085. (HEMRAJ) :—If $S = \sum_0^\infty \left\{ \frac{(2n)!}{2^{2n}(n!)^2} \quad \frac{1}{(2n+2)^4} \right\}$,

and $$\sigma = \sum_0^\infty \left\{ \frac{2n!}{2^{2n}(n!)^2} \cdot \frac{1}{(2n+1)^4} \right\},$$

find the value of $S + \frac{2}{\pi}\sigma$.

1086. (HEMRAJ) :—Evaluate

$$\int_0^1 \left[\frac{1}{x}\int_0^x \frac{\sin^{-1}x}{x}\,dx\right] dx.$$

1087. (A. A. KRISHNASWAMI AIYANGAR) :—ABC is any triangle. DEF is a triangle inscribed within it, D, E, F lying on the sides BC, CA, AB, respectively. O_1, O_2, O_3 are the orthocentres of the triangles AEF, BDF, CED.

Prove that (i) the area of the triangle $O_1O_2O_3$ is equal to that of the triangle DEF and (ii) the two triangles $O_1O_2O_3$ and DEF can be inscribed in the same conic.

1088. (A. A. KRISHNASWAMI AIYANGAR) :—Prove that the orthopole of one of the asymptotes of a rectangular hyperbola with reference to any inscribed triangle is the foot of the perpendicular from the orthocentre of the triangle on the other asymptote.

1088. (S. MAHADEVAN) :—Prove that if $|x| < 1$

$$\frac{x}{(1+x)^2}+\frac{x^2}{(1+x^2)^2}+\frac{x^3}{(1+x^3)^2}+\ldots\ldots$$

$$=\frac{x}{(1-x)^2}-\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}-\ldots\ldots=\sum_1^\infty \psi(n)x^{n^2}.$$

where $\psi(n)$ denotes the excess of the odd over the even divisors of n, 1 and n being reckoned as divisors (Bromwich, Chapter III).

1090. (S. MAHADEVAN) :—The focus S is joined to any point P_1 on the ellipse, P_1S cuts the ellipse again, in Q_1. Q_1S' cuts the ellipse again in P_2; and so on. Show that SP_n ultimately tends to coincide with the major axis.

1091. (ENQUIRER) :—If a, b, c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles that can be drawn to touch these three, prove that

$$\frac{1}{r_1}-\frac{1}{r_2}=\frac{2}{a}+\frac{2}{b}+\frac{2}{c}.$$

1092. (ENQUIRER):—Find the relation existing between α, β, γ, δ, if

$$\int\frac{\alpha+\beta x+\gamma x^2+\delta x^3}{\sqrt{x}\ (1-x)}dx$$

does not depend on circular functions.

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